

A Stepwise Planned Approach to the Solution of Hilbert's Sixth Problem. I : Noncommutative Symplectic Geometry and Hamiltonian Mechanics

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These papers will cover an open-ended project aimed at the solution of Hilbert's sixth problem (concerning joint axiomatization of physics and probability theory) proposed to be constructed in the framework of an all-embracing mechanics. In this first paper, the bare skeleton of such a mechanics is constructed in the form of noncommutative Hamiltonian mechanics which combines elements of noncommutative symplectic geometry and noncommutative probability in a super-algebraic setting. Canonically induced symplectic structure on the (skew) tensor product of two symplectic superalgebras (needed in the description of interaction between systems) is shown to exist if and only if either both system superalgebras are supercommutative or both non-supercommutative with a 'quantum symplectic structure' characterized by a *universal* Planck type constant; the presence of such a universal constant is, therefore, *dictated* by the formalism. This provides proper foundation for an autonomous development of quantum mechanics as a universal mechanics.

KEYWORDS: Hilbert's sixth problem; noncommutative Hamiltonian mechanics; noncommutative symplectic geometry; axiomatization of physics.

*To construct a sensible theory
Of all phenomena
You must unify
Probability and dynamics.*

I. INTRODUCTION

In the closing year of the nineteenth century, David Hilbert presented, in his famous Paris address [50], a list of 23 problems; the key statement of his sixth problem (henceforth called Hilprob6), presented under the title ‘Mathematical Treatment of the Axioms of Physics’, reads :

“To treat in the same manner, by means of axioms, the physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.”

A solution of this problem is expected to be an axiomatized version of a ‘theory of everything’ (TOE); this is more clear in the following reformulation of Hilprob6 :

“To evolve an axiomatic scheme covering all physics including the probabilistic framework employed for the treatment of statistical aspects of physical phenomena.”

Wightman’s article [73] is a decent review of the work relating to the solution of Hilprob6 up to mid-seventies. It covers Hilbert’s own work in this connection and the main developments relating to the axiomatization of quantum mechanics (QM) and of quantum field theory (QFT). Highlights in his treatment of axiomatization of QM are: von Neumann’s Hilbert space based axiomatics; Algebraic generalization of QM by Jordan, von Neumann and Wigner; Segal’s postulates for general quantum mechanics; Variants of the quantum logic approach by Birkhoff, von Neumann, Mackey, Piron and others; Hidden variable theories; EPR paradox and the question of completeness of QM. The highlights in the axiomatization of QFT are : Perturbative S-matrix and renormalization; Fock space; Reduction formulas; Representations of the inhomogeneous Lorentz group and relativistic wave equations; Haag’ theorem; Wightman formalism; PCT and Spin-statistics theorems in axiomatic field theory; Haag-Ruelle collision theory; C^* -algebraic approach to quantum field theory and statistical mechanics; Constructive field theory. Detailed references may be found in Wightman’s paper.

Developments during the past four decades, relevant to the present theme, relate to the following approaches : (i) Traditional gauge models (Weinberg [72]; Donoghue et al [28]), (ii) Superstring theory (Green, Schwarz and Witten [41]; Polchinski [60]), (iii) Loop quantum gravity (LQG)(Ashtekar and Lewandowski [5]; Rovelli [61]), (iv) Noncommutative geometry (NCG) (Connes [18]; Dubois-Violette [30]; Gracia-Bondia, Varilly and Figuerra [40]; Madore [56]; Landi [55]) based formalism for foundations of geometry (Connes [19,20]) and fundamental interactions (Chamseddine and Connes [16,17]; Connes and Marcolli [21]), (v) Local quantum physics or algebraic QFT (Haag [46]; Araki [3]; Horuzhy [51]; Bratteli and Robinson [12]; Borchers [9]), and (vi) Constructive field theory (Glimm and Jaffe [39]; Baez, Segal and Zhou [6]). Each of these approaches has its successes and problems. One expects that, in due course of time, these approaches will converge to an ‘all-embracing’ formalism which will eventually lead to a TOE which, with appropriate axiomatization, will yield a solution of Hilprob6.

Among the six approaches above, (v) has yet to accommodate concrete gauge models, whereas (vi) has yet to show concrete progress in four space-time dimensions. A common feature of the first four is that they are *quantization* programmes [i.e. one first develops the formalism in a classical setting (which may involve NCG) and then quantizes it]. The author considers this a drawback worth taking seriously and believes that, if one works, instead, in a framework which adopts a quantum outlook from the beginning, at least some of the Problems (those with a capital ‘P’; for example, that of providing, in a coherent scheme, a rationale for the choice of gauge group in gauge models) will be easier to solve. [To appreciate this point better, consider the chances of obtaining PCT and spin-statistics theorems by someone employing, instead of the manifestly covariant formalism of quantum field theory, one in which one only has tricks to *relativize* nonrelativistic equations.] The present series of papers is aimed at developing such a complementary approach (to the construction of the ‘all-embracing’ formalism) in which commitment to an autonomous quantum theoretic treatment is one of the top priorities.

The ‘all-embracing’ formalism must provide for a satisfactory treatment of the dynamics of the universe and its subsystems. Since all physics is essentially mechanics (every branch of physics deals with the dynamics of one or the other class of systems), the desired formalism must be an elaborate scheme of mechanics (with elements of probability incorporated). The commitment mentioned above implies that such a scheme of mechanics must incorporate, at least as a subdiscipline or in some approximation, an ambiguity-

free autonomous development of QM . One expects that, such a development will, starting with some appealing basics, connect smoothly to the traditional Hilbert space QM and facilitate a satisfactory treatment of quantum-classical correspondence and measurements.

For identifying appropriate ingredients for the desired formalism, one must go back to physics basics. Physics is concerned with observations, studying correlations between observations, theorizing about those correlations and making theory-based conditional predictions/retrodictions about observations. The desired formalism must, therefore, incorporate a rich enough description of observations so as to adequately cover all these aspects. We are accustomed to describing observations as geometrical facts. The formalism must, therefore, have an all-embracing underlying geometry. An appealing choice for the same is NCG which is known to cover traditional/commutative differential geometry as a special case. It has, moreover, excellent consonance with our theme because noncommutativity is the hallmark of QM. Indeed, the central point made in Heisenberg's paper [48] that marked the birth of QM was that the kinematics underlying QM must be based on a non-commutative algebra of observables. This idea was developed into a scheme of mechanics — called matrix mechanics — by Born, Jordan, Dirac and Heisenberg [11,26,10]. The proper geometrical framework for the construction of the quantum Poisson brackets of matrix mechanics is provided by non-commutative symplectic structures treated by Dubois-Violette and coworkers [30,33,31,32]. The NCG scheme employed in these works (referred to henceforth as DVNCG) is a straightforward generalization of the scheme of commutative differential geometry in which the algebra $C^\infty(M)$ of smooth complex-valued functions on a manifold M is replaced by a general (not necessarily commutative) complex associative $*$ -algebra \mathcal{A} and the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M by the Lie algebra $\text{Der}(\mathcal{A})$ of derivations of \mathcal{A} .

The key observation underlying the present work is the fact that the $*$ -algebras of the type employed in DVNCG also provide a general framework for an observable-state based treatment of quantum probability (Meyer [58]). This allows us to adopt the strategy of combining elements of noncommutative symplectic geometry and noncommutative probability in an algebraic framework; this promises to be a reasonably deep level realization of unification of probability and dynamics (advocated in the poem above) in the true spirit of Hilprob6.

The scheme based on normed algebras (Jordan, von Neumann and Wigner

[53]; Segal [66,67]; Haag and Kastler [47]; Haag [46]; Araki [3]; Emch [36,37]; Bratteli and Robinson [12]), although it has an observable - state framework of the type mentioned above, does not serve our needs because it is not suitable for a sufficiently general treatment of noncommutative symplectic geometry. Iguri and Castagnino [52] have analyzed the prospects of a more general class of topological algebras (nuclear, barreled b^* -algebras) as a mathematical framework for the formulation of quantum principles prospectively better than that of the normed algebras. These algebras accommodate unbounded observables at the abstract level. Following essentially the ‘footsteps’ of Segal [66], they obtain some results parallel to those in the C^* -algebra theory — an extremal decomposition theorem for states, a functional representation theorem for commutative subalgebras of observables and an extension of the classical GNS theorem. A paper by Dimakis and Müller-Hoissen [25] is devoted to constructing consistent differential calculi on the algebra generated by operators satisfying the canonical commutation relations and studying invariance properties of the resulting constructions. These works are essentially complementary to the present one where the emphasis is on the development of noncommutative *Hamiltonian mechanics*.

Development of such a mechanics in the algebraic framework requires some augmentation and generalization of DVNCG in the following respects :

- (i) One needs the noncommutative analogues of the push-forward and pull-back mappings induced by diffeomorphisms between manifolds on vector fields and differential forms (which play important roles in classical symplectic geometry).
- (ii) It is desirable to have a generalization of DVNCG based on algebraic pairs $(\mathcal{A}, \mathcal{X})$ where \mathcal{A} is a $*$ -algebra as above and \mathcal{X} is a Lie subalgebra of $\text{Der}(\mathcal{A})$. (This is noncommutative analogue of working on a leaf of a foliation; see section II D.) Such a generalization will be needed in the development of noncommutative symplectic geometry with sufficient generality and in the algebraic treatment of general quantum systems.

The formalism being aimed at is expected to have Wightman’s axiomatic formalism (involving Bose as well as Fermi fields) and appropriate algebraic versions of modern gauge theories and superstring theories as special cases. This necessitates accommodating fermionic objects on an equal footing with the bosonic ones; to achieve this, we shall employ superalgebras as the basic objects.

In the next section, a superalgebraic version of DVNCG is presented which incorporates the improvement in the definition of noncommutative differen-

tial forms introduced in [31,32] [i.e. demanding $\omega(\dots, KX, \dots) = K\omega(\dots, X, \dots)$ where K is in the center of the algebra; for notation, see section 2.2] and the augmentation and generalization of DVNCG mentioned above. In section III, a straightforward development of noncommutative symplectic geometry and noncommutative Hamiltonian mechanics (NHM) is presented. It is in the form of an observable-state based algebraic treatment of mechanics parallel to that of classical statistical mechanics; the concept of a classical Hamiltonian system is generalized to that of an NHM Hamiltonian system. Induced mappings on differential forms mentioned above play an important role in a systematic treatment of symplectic mappings and invariance principles and in defining equivalence of NHM Hamiltonian systems. This section also includes a treatment, in the NHM framework, of symplectic actions of Lie groups and noncommutative generalizations of the momentum map, Poincare-Cartan form and the symplectic version of Noether's theorem [Theorem(1)].

Section IV contains the treatment of two interacting systems in the framework of NHM. An important result obtained there [Theorem (2)] is that, given two systems represented by symplectic superalgebras $\mathcal{A}^{(i)}$ ($i = 1, 2$) with symplectic forms $\omega^{(i)}$, the ‘canonically induced’ two-form ω [given by Eq.(78) below] on the (skew) tensor product $\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ (the system algebra of the coupled system) represents a genuine symplectic structure if and only if either both the superalgebras are supercommutative or both non-supercommutative with a ‘quantum symplectic structure’ [i.e. one which gives a Poisson bracket which is a (super-)commutator up to multiplication by a constant (ih_0^{-1}) where h_0 is a real-valued constant of the dimension of action] characterized by a *universal* parameter h_0 . It follows that the formalism, firstly, prohibits a ‘quantum-classical interaction’, and, secondly, has a natural place for the Planck constant as a universal constant — just as special relativity has a natural place for a universal speed. In fact, the situation here is somewhat better because whereas, in special relativity, the existence of a universal speed is *postulated*, in NHM, the existence of a universal Planck-like constant is *dictated/predicted* by the formalism of NHM.

The last section contains some concluding remarks.

The formalism of NHM needs to be augmented to make a satisfactory autonomous treatment of quantum systems possible; this augmentation and an autonomous development of QM in the framework of augmented NHM will be presented in paper II; in the third paper, the formalism evolved in I and II will be applied to the treatment of quantum measurements and obtain

a straightforward derivation of the von Neumann projection/collapse rule. In subsequent papers, it is planned to treat systems with dynamical symplectic structure (noncommutative Lagrangian systems) and to develop consistent treatments of the dynamics of space-time geometry and fields (aiming at a satisfactory unified formalism for fundamental interactions) satisfying the criterion mentioned above.

II. SUPERDERIVATION-BASED DIFFERENTIAL CALCULUS

In this section, we present essential developments in the noncommutative differential calculus to be employed in this series of papers. This is a superalgebraic version of DVNCG augmented and generalized as mentioned above. Topics covered include some not so well known results about superalgebras and superderivations, noncommutative differential forms and some of their transformation theory and superderivations and differential forms on (skew) tensor products of superalgebras.

Some good references for the background material for this section are (apart from the references for DVNCG given above; see also the review article by Djemai [27]) (Greub [42,43]; Pittner [59]; Giachetta, Mangiarotti and Sardanshvily [38]; Scheunert [63,64]; Scheunert and Zhang [65]).

Note. In most applications, the non-super version of the formalism developed below is adequate; this can be obtained by simply putting, in the formulas below, all the epsilons representing parities equal to zero.

A. Superalgebras and superderivations

We first recall a few basic concepts in superalgebra. A *supervector space* is a (complex) vector space $V = V^{(0)} \oplus V^{(1)}$; a vector $v \in V$ can be uniquely expressed as a sum $v = v_0 + v_1$ of even and odd vectors; they are assigned parities $\epsilon(v_0) = 0$ and $\epsilon(v_1) = 1$. Objects with definite parity are called homogeneous. We shall denote the parity of a homogeneous object w by $\epsilon(w)$ or ϵ_w according to convenience. A *superalgebra* \mathcal{A} is a supervector space which is an associative algebra with identity (denoted as $I_{\mathcal{A}}$ or simply I); it becomes a **-superalgebra* if an antilinear *-operation or involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ is defined such that $(A^*)^* = A$, $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$, $I^* = I$ and, for homogeneous $A \in \mathcal{A}$, $\epsilon(A^*) = \epsilon(A)$. An element $A \in \mathcal{A}$ is called *hermitian* if $A^* = A$.

Note. In a couple of earlier versions of this paper (arXiv : 0909.4606 v1, v2), the following convention (following [32]) was adopted : For homogeneous A and B, $(AB)^* = \eta_{AB}B^*A^*$ where $\eta_{AB} = (-1)^{\epsilon_A\epsilon_B}$. This convention, however,

does not suit our needs. For example, given two fermionic annihilation operators a, b , we have, in traditional quantum field theory, $(ab)^* = b^*a^*$ and not $(ab)^* = -b^*a^*$.

The *supercommutator* of two homogeneous elements A, B of \mathcal{A} is defined as $[A, B] = AB - \eta_{AB}BA$; the definition is extended to general elements by bilinearity. We shall also employ the notations $[A, B]_{\mp} = AB \mp BA$. A superalgebra \mathcal{A} is said to be *supercommutative* if the supercommutator of every pair of its elements vanishes. The *graded center* of \mathcal{A} , denoted as $Z(\mathcal{A})$, consists of those elements of \mathcal{A} which have vanishing supercommutators with all elements of \mathcal{A} ; it is clearly a supercommutative superalgebra. A *Lie superalgebra* is a supervector space \mathcal{L} with a *superbracket* operation $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ which is (i) bilinear, (ii) graded skew-symmetric which means that, for any two homogeneous elements $a, b \in \mathcal{L}$, $[a, b] = -\eta_{ab}[b, a]$ and (iii) satisfies the *super Jacobi identity*

$$[a, [b, c]_{\cdot}] = [[a, b]_{\cdot}, c] + \eta_{ab}[b, [a, c]_{\cdot}] \text{ for all } a, b, c \in \mathcal{L}.$$

[The subscript dot (\cdot) has been temporarily introduced to keep this object distinct from the supercommutator.]

We shall generally employ topological algebras [29,49] — complete, Hausdorff, separable locally convex algebras with a jointly continuous product (with a continuous star operation and Z_2 -grading); motivations for this choice are described in section III D. Mappings involving topological spaces should be understood as continuous unless stated otherwise.

A *(*)homomorphism* of a superalgebra \mathcal{A} into \mathcal{B} is a linear mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ which preserves products, identity elements, parities (and involutions) :

$$\Phi(AB) = \Phi(A)\Phi(B), \quad \Phi(I_{\mathcal{A}}) = I_{\mathcal{B}}, \quad \epsilon(\Phi(A)) = \epsilon(A), \quad \Phi(A^*) = (\Phi(A))^*;$$

if it is, moreover, bijective, it is called a *(*)isomorphism*.

A (homogeneous) *superderivation* of a superalgebra \mathcal{A} is a linear map $X : \mathcal{A} \rightarrow \mathcal{A}$ such that $X(AB) = X(A)B + \eta_{XA}AX(B)$ for all homogeneous A and all B in \mathcal{A} . The *multiplication operator* μ on \mathcal{A} associates, with every $A \in \mathcal{A}$, a linear mapping $\mu(A) : \mathcal{A} \rightarrow \mathcal{A}$ such that $\mu(A)B = AB$ for all $B \in \mathcal{A}$.

Proposition 2.1. *Given a superalgebra \mathcal{A} , a necessary and sufficient condition that a linear map $X : \mathcal{A} \rightarrow \mathcal{A}$ is a homogeneous superderivation is*

$$X \circ \mu(A) - \eta_{XA} \mu(A) \circ X = \mu(X(A)) \text{ for all homogeneous } A \in \mathcal{A}. \quad (1)$$

Proof. Eq.(1) gives, on making each side act on a general element B of \mathcal{A} , precisely the equation defining a homogeneous superderivation X above. \square

The space $\text{SDer}(\mathcal{A}) [= \text{SDer}(\mathcal{A})^{(0)} \oplus \text{SDer}(\mathcal{A})^{(1)}]$ of all superderivations of \mathcal{A} constitutes a Lie superalgebra with $[X, Y] = [X, Y]$. The *inner superderivations* D_A defined by $D_A B = [A, B]$ satisfy the relation $[D_A, D_B] = D_{[A, B]}$ and constitute a Lie sub-superalgebra $\text{ISDer}(\mathcal{A})$ of $\text{SDer}(\mathcal{A})$.

As in DVNCG, it will be implicitly assumed that the superalgebras being employed have a reasonably rich supply of superderivations so that various constructions involving them have a nontrivial content. Some discussion and useful results relating to the precise characterization of the relevant class of algebras may be found in [35].

Proposition 2.2. *Given a superalgebra \mathcal{A} , $K \in Z(\mathcal{A})$ and $X, Y \in \text{SDer}(\mathcal{A})$, we have (i) $X(K) \in Z(\mathcal{A})$, (ii) $KX \in \text{SDer}(\mathcal{A})$ [with $(KX)(A) \equiv K[X(A)]$ for all $A \in \mathcal{A}$], and (iii) with X, K homogeneous, the relation*

$$[X, KY] = X(K)Y + \eta_{XK}K[X, Y]. \quad (2)$$

Proof. Easily verified. \square

An involution $*$ on $\text{SDer}(\mathcal{A})$ is defined by the relation $X^*(A) = [X(A^*)]^*$.

Proposition 2.3. *In obvious notation*

(i) $[X, Y]^* = [X^*, Y^*]$; (ii) $(D_A)^* = -D_{A^*}$.

Proof. Easily verified. \square

A superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a linear mapping

$$\Phi_* : \text{SDer}(\mathcal{A}) \rightarrow \text{SDer}(\mathcal{B}) \text{ given by } (\Phi_* X)(B) = \Phi(X[\Phi^{-1}(B)]) \quad (3)$$

for all $X \in \text{SDer}(\mathcal{A})$ and $B \in \mathcal{B}$. It is the analogue (and a generalization) of the push-forward mapping induced by a diffeomorphism between two manifolds on the vector fields.

Proposition 2.4. *With Φ as above and $\Psi : \mathcal{B} \rightarrow \mathcal{C}$ we have*

$$\begin{aligned} (i) \quad & (\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*; \quad (ii) \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y]; \\ (iii) \quad & [\Phi_*(X)]^* = \Phi_*(X^*). \end{aligned} \quad (4)$$

Proof. (i) For any $X \in \text{SDer}(\mathcal{A})$ and $C \in \mathcal{C}$,

$$\begin{aligned} [(\Psi \circ \Phi)_* X](C) &= (\Psi \circ \Phi)(X[(\Psi \circ \Phi)^{-1}(C)]) = \Psi[\Phi(X[\Phi^{-1}(\Psi^{-1}(C))])] \\ &= \Psi[(\Phi_* X)(\Psi^{-1}(C))] = [\Psi_*(\Phi_* X)](C). \end{aligned}$$

(ii) It is clearly adequate to prove it for homogeneous $X, Y \in SDer(\mathcal{A})$. For any $B \in \mathcal{B}$, we have

$$\begin{aligned} (\Phi_*[X, Y])(B) &= \Phi([X, Y](\Phi^{-1}(B))) \\ &= \Phi[X(Y[\Phi^{-1}(B)])] - \eta_{XY}\Phi[Y(X[\Phi^{-1}(B)])]. \end{aligned}$$

Now, inserting $\Phi^{-1} \circ \Phi$ between X and Y in each of the two terms on the right, the right hand side is easily seen to be

$$(\Phi_*X)[(\Phi_*Y)(B)] - \eta_{XY}(\Phi_*Y)[(\Phi_*X)(B)].$$

(iii) For any $B \in \mathcal{B}$, we have

$$\begin{aligned} [\Phi_*(X)]^*(B) &= [\Phi_*(X)(B^*)]^* = [\Phi(X[\Phi^{-1}(B^*)])]^* \\ &= \Phi[(X[\Phi^{-1}(B^*)])^*] = \Phi(X^*[\Phi^{-1}(B)]) = (\Phi_*X^*)(B) \quad \square \end{aligned}$$

Corollary 2.5. *The mapping Φ_* defined by Eq.(3) is an involution preserving Lie superalgebra isomorphism.*

B. Noncommutative differential forms

Main developments in this subsection are parallel to those in (Grosse and Reiter [44]) who have generalized the treatment of differential geometry of matrix algebras in [33] to supermatrix algebras. Some related work on supermatrix geometry has also appeared in [34,54].

The mathematical object employed in DVNCG [31] to define a differential calculus is a graded differential $*$ -algebra which is a graded differential algebra (\mathcal{U}, d) where $\mathcal{U} = \bigoplus_{k \geq 0} \mathcal{U}^k$ is a graded algebra (with product $\mathcal{U}^j \mathcal{U}^k \subset \mathcal{U}^{j+k}$) and d is the differential of \mathcal{U} (antiderivation of degree 1 satisfying the condition $d^2 = 0$) equipped with an antilinear involution $\omega \mapsto \omega^*$ which preserves degree and satisfies the conditions

- (i) $(\alpha.\beta)^* = (-1)^{jk}\beta^*.\alpha^*$ for all $\alpha \in \mathcal{U}^j, \beta \in \mathcal{U}^k$;
- (ii) $(d\omega)^* = d(\omega^*)$ for all $\omega \in \mathcal{U}$.

Given a complex unital $*$ -algebra \mathcal{A} (not necessarily commutative), a *differential calculus over \mathcal{A}* is a graded differential $*$ -algebra as above with $\mathcal{U}^0 = \mathcal{A}$. The traditional differential form calculus on a manifold M is a special case of this with $\mathcal{A} = C^\infty(M)$, \mathcal{U}^k the space of differential k -forms on M , the exterior product as product and exterior derivative as the differential.

We shall construct a differential calculus on a superalgebra \mathcal{A} by employing the Chevalley-Eilenberg cochains [15,71,38]. We recall the relevant basic definitions below.

Let \mathcal{G} be a Lie algebra over the field \mathbb{K} [which may be \mathbb{R} or \mathbb{C}] and V a \mathcal{G} -module which means it is a vector space over \mathbb{K} having defined on it a \mathcal{G} -action associating a linear mapping $\Psi(\xi)$ on V with every element ξ of \mathcal{G} such that

$$\Psi(0) = id_V \quad \text{and} \quad \Psi([\xi, \eta]) = \Psi(\xi) \circ \Psi(\eta) - \Psi(\eta) \circ \Psi(\xi) \quad \text{for all } \xi, \eta \in \mathcal{G}$$

where id_V is the identity mapping on V . A V -valued p -cochain $\lambda^{(p)}$ of \mathcal{G} ($p = 1, 2, \dots$) is a skew-symmetric multilinear map from \mathcal{G}^p into V . These cochains constitute a vector space $C^p(\mathcal{G}, V)$; one defines $C^0(\mathcal{G}, V) = V$. The coboundary operator $d : C^p(\mathcal{G}, V) \rightarrow C^{p+1}(\mathcal{G}, V)$ defined by

$$\begin{aligned} (d\lambda^{(p)})(\xi_0, \xi_1, \dots, \xi_p) &= \sum_{i=0}^p (-1)^i \Psi(\xi_i) [\lambda^{(p)}(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_p)] + \\ &\quad \sum_{0 \leq i < j \leq p} (-1)^j \lambda^{(p)}(\xi_0, \dots, \xi_{i-1}, [\xi_i, \xi_j], \xi_{i+1}, \dots, \hat{\xi}_j, \dots, \xi_p) \end{aligned} \quad (5)$$

(the hat indicates omission) for $\xi_0, \dots, \xi_p \in \mathcal{G}$ satisfies the condition $d^2 = 0$. Defining

$$C(\mathcal{G}, V) = \bigoplus_{p \geq 0} C^p(\mathcal{G}, V),$$

the pair $(C(\mathcal{G}, V), d)$ constitutes a cochain complex. The subspaces of $C^p(\mathcal{G}, V)$ consisting of closed cochains (cocycles) [i.e. those λ^p satisfying $d\lambda^p = 0$] and exact cochains (coboundaries) [i.e. those λ^p satisfying $\lambda^p = d\mu^{p-1}$ for some $(p-1)$ -cochain μ^{p-1}] are denoted as $Z^p(\mathcal{G}, V)$ and $B^p(\mathcal{G}, V)$ respectively; the quotient space $H^p(\mathcal{G}, V) \equiv Z^p(\mathcal{G}, V)/B^p(\mathcal{G}, V)$ is called the p -th cohomology module of \mathcal{G} with coefficients in V .

For the special case of the trivial action of \mathcal{G} on V [i.e. $\Psi(\xi) = 0 \quad \forall \xi \in \mathcal{G}$], a subscript zero is attached to these spaces [$C_0^p(\mathcal{G}, V)$ etc]. In this case, we record, for future use, the form Eq.(5) takes for $p = 1, 2$:

$$\begin{aligned} d\lambda^{(1)}(\xi_0, \xi_1) &= -\lambda^{(1)}([\xi_0, \xi_1]) \\ d\lambda^{(2)}(\xi_0, \xi_1, \xi_2) &= -[\lambda^{(2)}([\xi_0, \xi_1], \xi_2) + \text{cyclic terms in } \xi_0, \xi_1, \xi_2]. \end{aligned} \quad (6)$$

Recalling that the classical differential p -forms on a manifold M are defined as skew-symmetric multilinear maps of $\mathcal{X}(M)^p$ into $C^\infty(M)$, the

De Rham complex of classical differential forms can be seen as a special case of Chevalley-Eilenberg complex with $\mathcal{G} = \mathcal{X}(M)$, $V = C^\infty(M)$ and $\Psi(X)(f) = X(f)$ in obvious notation. [For the relevant algebraic definition of the classical exterior derivative, see Matsushima [57], p. 140; it is Eq.(5) above with $\lambda^{(p)}$ a traditional differential p-form and ξ_j s vector fields.] Replacing $C^\infty(M)$ by a superalgebra [complex, associative, unital (i.e. possessing a unit element), not necessarily supercommutative] \mathcal{A} and $\mathcal{X}(M)$ by $SDer(\mathcal{A})$ [so that $\Psi(X)(A) = X(A)$ for all $X \in SDer(\mathcal{A})$ and $A \in \mathcal{A}$], a natural choice for the space of noncommutative differential p-forms is the space of \mathcal{A} -valued p-cochains of $SDer(\mathcal{A})$:

$$C^p(SDer\mathcal{A}, \mathcal{A}) [= C^{p,0}(SDer(\mathcal{A}), \mathcal{A}) \oplus C^{p,1}(SDer(\mathcal{A}), \mathcal{A})]$$

with $C^0(SDer(\mathcal{A}), \mathcal{A}) = \mathcal{A}$. For $\omega \in C^{p,s}(SDer(\mathcal{A}), \mathcal{A})$, and homogeneous $X, Y \in SDer(\mathcal{A})$, we have

$$\omega(..., X, Y, ...) = -\eta_{XY}\omega(..., Y, X, ...). \quad (7)$$

For a general permutation σ of the arguments of ω , we have

$$\omega(X_{\sigma(1)}, ..., X_{\sigma(p)}) = \kappa_\sigma \gamma_p(\sigma; \epsilon_{X_1}, ..., \epsilon_{X_p}) \omega(X_1, ..., X_p)$$

where κ_σ is the parity of the permutation σ and

$$\gamma_p(\sigma; s_1, ..., s_p) = \prod_{\substack{j, k = 1, ..., p; \\ j < k, \sigma^{-1}(j) > \sigma^{-1}(k)}} (-1)^{s_j s_k}.$$

The *exterior product*

$$\wedge : C^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \times C^{q,s}(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C^{p+q, r+s}(SDer(\mathcal{A}), \mathcal{A})$$

is defined as

$$(\alpha \wedge \beta)(X_1, ..., X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} \kappa_\sigma \gamma_{p+q}(\sigma; \epsilon_{X_1}, ..., \epsilon_{X_{p+q}}) (-1)^{s \sum_{j=1}^p \epsilon_{X_{\sigma(j)}}} \alpha(X_{\sigma(1)}, ..., X_{\sigma(p)}) \beta(X_{\sigma(p+1)}, ..., X_{\sigma(p+q)}).$$

With this product, the graded vector space

$$C(SDer(\mathcal{A}), \mathcal{A}) = \bigoplus_{p \geq 0} C^p(SDer(\mathcal{A}), \mathcal{A})$$

becomes an $\mathbb{N}_0 \times Z_2$ -bigraded complex algebra. (Here \mathbb{N}_0 is the set of non-negative integers.)

An involution $*$: $C^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C^{p,r}(SDer(\mathcal{A}), \mathcal{A})$ is defined by the relation $\omega^*(X_1, \dots, X_p) = [\omega(X_1^*, \dots, X_p^*)]^*$; ω is said to be real (imaginary) if $\omega^* = \omega(-\omega)$. We have, for α, β as above,

$$(\alpha \wedge \beta)^* = (-1)^{pq} \beta^* \wedge \alpha^*.$$

[Hint. One expects $(\alpha \wedge \beta)^* = \lambda \beta^* \wedge \alpha^*$ where λ can possibly depend on p, q, r, s and takes values ± 1 . Applying the $*$ -operation on the equation for $\alpha \wedge \beta$ with X_1, \dots, X_{p+q} even (parities of the superderivations not being relevant) and using the definition of ω^* above, a simple combinatorial argument gives the desired result. The fact that λ is independent of r and s is related to the fact that the chosen convention for the $*$ -operation on the superalgebra \mathcal{A} does not involve parities.]

The Lie superalgebra $SDer(\mathcal{A})$ acts on itself and on $C(SDer(\mathcal{A}), \mathcal{A})$ through *Lie derivatives*. For each $Y \in SDer(\mathcal{A})^{(r)}$, one defines linear mappings $L_Y : SDer(\mathcal{A})^{(s)} \rightarrow SDer(\mathcal{A})^{(r+s)}$ and $L_Y : C^{p,s}(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C^{p,r+s}(SDer(\mathcal{A}), \mathcal{A})$ such that the following three conditions hold (with obvious notation):

$$\begin{aligned} L_Y(A) &= Y(A); \quad L_Y[X(A)] = (L_Y X)(A) + \eta_{XY} X[L_Y(A)]; \\ L_Y[\omega(X_1, \dots, X_p)] &= (L_Y \omega)(X_1, \dots, X_p) + \sum_{i=1}^p (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + \dots + \epsilon_{X_{i-1}})} \\ &\quad \cdot \omega(X_1, \dots, X_{i-1}, L_Y X_i, X_{i+1}, \dots, X_p). \end{aligned}$$

The first two conditions give

$$L_Y X = [Y, X]$$

which, along with the third, gives

$$\begin{aligned} (L_Y \omega)(X_1, \dots, X_p) &= Y[\omega(X_1, \dots, X_p)] - \sum_{i=1}^p (-1)^{\epsilon_Y(\epsilon_\omega + \epsilon_{X_1} + \dots + \epsilon_{X_{i-1}})} \\ &\quad \cdot \omega(X_1, \dots, X_{i-1}, [Y, X_i], X_{i+1}, \dots, X_p). \end{aligned} \quad (8)$$

Two important properties of the Lie derivative are, in obvious notation,

$$\begin{aligned} [L_X, L_Y] &= L_{[X, Y]} \\ L_Y(\alpha \wedge \beta) &= (L_Y \alpha) \wedge \beta + \eta_{Y\alpha} \alpha \wedge (L_Y \beta). \end{aligned}$$

For each $X \in SDer(\mathcal{A})^{(r)}$, we define the *interior product*
 $i_X : C^{p,s}(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C^{p-1,r+s}(SDer(\mathcal{A}), \mathcal{A})$ (for $p \geq 1$) by

$$(i_X \omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1}).$$

One defines $i_X(A) = 0$ for all $A \in \mathcal{A} \equiv C^0(SDer(\mathcal{A}), \mathcal{A})$. Some important properties of the interior product are :

$$i_X \circ i_Y + \eta_{XY} i_Y \circ i_X = 0;$$

$$(L_Y \circ i_X - i_X \circ L_Y) \omega = \eta_{Y\omega} i_{[Y,X]} \omega. \quad (9)$$

The *exterior derivative* $d : C(SDer(\mathcal{A}), \mathcal{A}) \rightarrow C(SDer(\mathcal{A}), \mathcal{A})$ such that $dC^{p,r}(SDer(\mathcal{A}), \mathcal{A}) \subset C^{p+1,r}(SDer(\mathcal{A}), \mathcal{A})$ is defined as follows :

$$\begin{aligned} (d\omega)(X_0, X_1, \dots, X_p) &= \sum_{i=0}^p (-1)^{i+a_i} X_i [\omega(X_0, \dots, \hat{X}_i, \dots, X_p)] \\ &+ \sum_{0 \leq i < j \leq p} (-1)^{j+b_{ij}} \omega(X_0, \dots, X_{i-1}, [X_i, X_j], X_{i+1}, \dots, \hat{X}_j, \dots, X_p) \end{aligned} \quad (10)$$

where

$$a_i = \epsilon_{X_i}(\epsilon_\omega + \sum_{k=0}^{i-1} \epsilon_{X_k}); \quad b_{ij} = \epsilon_{X_j} \sum_{k=i+1}^{j-1} \epsilon_{X_k}.$$

[Eq.(10) is clearly a special case of Eq.(5).] It satisfies the relations (i) $d^2(= d \circ d) = 0$, (ii) $d(\omega^*) = (d\omega)^*$, (iii) $d \circ L_Y = L_Y \circ d$, and (iv) the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$$

where α is a p -cochain. The first of these equations shows that the pair $(C(SDer(\mathcal{A}), \mathcal{A}), d)$ constitutes a cochain complex. We also have the important relation

$$(i_X \circ d + d \circ i_X) \omega = \eta_{X\omega} L_X \omega. \quad (11)$$

Taking clue from [31,32] (where the subcomplex of $Z(\mathcal{A})$ -linear cochains $[Z(\mathcal{A})$ being, in the notation of these papers, the center of the algebra $\mathcal{A}]$ was adopted as the space of differential forms), we consider the subset $\Omega(\mathcal{A})$

of $C(SDer(\mathcal{A}), \mathcal{A})$ consisting of $Z(\mathcal{A})$ -linear cochains [where $Z(\mathcal{A})$ is now the graded center of the superalgebra \mathcal{A}]. Eq.(2) ensures that this subset is closed under the action of d and, therefore, a subcomplex. We shall take this space to be the space of differential forms in subsequent geometrical work. We have, of course,

$$\Omega(\mathcal{A}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{A})$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^p(\mathcal{A}) = \Omega^{p,0}(\mathcal{A}) \oplus \Omega^{p,1}(\mathcal{A})$ for all $p \geq 0$.

The quadruple $(\Omega(\mathcal{A}), \wedge, d, *)$, with the three operations and their mutual relations as given above is an $\mathbb{N}_0 \times \mathbb{Z}_2$ -bigraded differential $*$ -algebra constituting a differential calculus over \mathcal{A} ; it is a ‘subcalculus’ of the larger differential calculus $(C(SDer(\mathcal{A}), \mathcal{A}), \wedge, d, *)$ over \mathcal{A} .

C. Induced mappings on differential forms

A superalgebra $*$ -isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces, besides the Lie superalgebra-isomorphism $\Phi_* : SDer(\mathcal{A}) \rightarrow SDer(\mathcal{B})$, a linear mapping

$$\Phi^* : C^{p,s}(SDer(\mathcal{B}), \mathcal{B}) \rightarrow C^{p,s}(SDer(\mathcal{A}), \mathcal{A})$$

given, for all $\omega \in C^{p,s}(SDer(\mathcal{B}), \mathcal{B})$ and $X_1, \dots, X_p \in SDer(\mathcal{A})$, by

$$(\Phi^* \omega)(X_1, \dots, X_p) = \Phi^{-1}[\omega(\Phi_* X_1, \dots, \Phi_* X_p)]. \quad (12)$$

The mapping Φ preserves (bijectively) all the algebraic relations. Eq.(3) shows that Φ_* preserves $Z(\mathcal{A})$ -linear combinations of the superderivations. It follows that Φ^* maps differential forms (with their restricted definition given at the end of section II B) onto differential forms. These mappings are analogues (and generalizations) of the pull-back mappings on differential forms (on manifolds) induced by diffeomorphisms.

Proposition 2.6. *With Φ and Ψ as in proposition 2.4, $\alpha \in C^{p,r}(SDer(\mathcal{B}), \mathcal{B})$ and $\beta \in C^{q,s}(SDer(\mathcal{B}), \mathcal{B})$, we have*

$$(i) (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*; \quad (ii) \Phi^*(\alpha \wedge \beta) = (\Phi^* \alpha) \wedge (\Phi^* \beta); \quad (13)$$

$$(iii) \Phi^*(d\alpha) = d(\Phi^* \alpha); \quad (14)$$

$$(iv) (\Phi^* \omega)^* = \Phi^*(\omega^*). \quad (15)$$

[Reader, please note the two different uses of $*$ in Eq.(15).]

Proof. (i) For $\omega \in C^{p,s}(SDer(\mathcal{C}), \mathcal{C})$ and $X_1, \dots, X_p \in SDer(\mathcal{A})$,

$$\begin{aligned} [(\Psi \circ \Phi)^* \omega](X_1, \dots, X_p) &= (\Phi^{-1} \circ \Psi^{-1})[\omega(\Psi_*(\Phi_* X_1), \dots, \Psi_*(\Phi_* X_p))] \\ &= \Phi^{-1}[(\Psi^* \omega)(\Phi_* X_1, \dots, \Phi_* X_p)] \\ &= [\Phi^*(\Psi^* \omega)](X_1, \dots, X_p). \end{aligned}$$

(ii) For $X_1, \dots, X_{p+q} \in SDer(\mathcal{A})$, we have

$$[\Phi^*(\alpha \wedge \beta)](X_1, \dots, X_{p+q}) = \Phi^{-1}[(\alpha \wedge \beta)(\Phi_* X_1, \dots, \Phi_* X_{p+q})].$$

Expanding the wedge product and noting that

$$\Phi^{-1}[\alpha(\cdot)\beta(\cdot)] = \Phi^{-1}[\alpha(\cdot)] \cdot \Phi^{-1}[\beta(\cdot)],$$

the right hand side is easily seen to be equal to $[(\Phi^* \alpha) \wedge (\Phi^* \beta)](X_1, \dots, X_{p+q})$.

(iii) We have

$$[\Phi^*(d\alpha)](X_0, \dots, X_p) = \Phi^{-1}[(d\alpha)(\Phi_* X_0, \dots, \Phi_* X_p)].$$

Using Eq.(10) for $d\alpha$ and noting that, in the first sum,

$$\begin{aligned} \Phi^{-1}[(\Phi_* X_i)(\alpha(\Phi_* X_0, \dots))] &= \Phi^{-1}[\Phi(X_i[\Phi^{-1}(\alpha(\Phi_* X_0, \dots))])] \\ &= X_i[(\Phi^* \alpha)(X_0, \dots)] \end{aligned}$$

and (recalling that $[\Phi_* X_i, \Phi_* X_j] = \Phi_*[X_i, X_j]$), in the second sum,

$$\begin{aligned} \Phi^{-1}[\alpha(\Phi_* X_0, \dots, \Phi_* X_i, \Phi_*[X_i, X_j], \dots, \Phi_* \hat{X}_j, \dots, \Phi_* X_p)] \\ (\Phi^* \alpha)(X_0, \dots, X_i, [X_i, X_j], \dots, \hat{X}_j, \dots, X_p), \end{aligned}$$

it is easily seen that the left hand side of Eq.(14), evaluated at (X_0, \dots, X_p) , equals $[(d(\Phi^* \alpha)](X_0, \dots, X_p)$.

(iv) With the notation in the proof of (i) above, we have

$$\begin{aligned} (\Phi^* \omega)^*(X_1, \dots, X_p) &= [(\Phi^* \omega)(X_1^*, \dots, X_p^*)]^* \\ &= (\Phi^{-1}[\omega(\Phi_* X_1^*, \dots, \Phi_* X_p^*)])^* \\ &= \Phi^{-1}[\omega((\Phi_* X_1)^*, \dots, (\Phi_* X_p)^*)]^* \\ &= \Phi^{-1}[\omega^*(\Phi_* X_1, \dots, \Phi_* X_p)] \\ &= (\Phi^* \omega^*)(X_1, \dots, X_p). \quad \square \end{aligned}$$

Corollary 2.7. *The mapping Φ^* defined by Eq.(12) is an isomorphism of the $\mathbb{N}_0 \times \mathbb{Z}_2$ -bigraded differential $*$ -algebras mapping $(C(SDer(\mathcal{B}), \mathcal{B}), \wedge, d, *)$ onto $(C(SDer(\mathcal{A}), \mathcal{A}), \wedge, d, *)$ restricting to an isomorphism of the similarly graded differential subalgebras mapping $(\Omega(\mathcal{B}), \wedge, d, *)$ onto $(\Omega(\mathcal{A}), \wedge, d, *)$.*

Taking \mathcal{A} to be a topological algebra as mentioned above, let $\Phi_t : \mathcal{A} \rightarrow \mathcal{A}$ ($t \in I_0$ where I_0 is an interval in \mathbb{R} containing the origin) be a one-parameter family of transformations (i.e. superalgebra isomorphisms) satisfying the conditions

$$\Phi_s \circ \Phi_t = \Phi_{s+t} \text{ for all } s, t, s+t \in I_0; \quad \Phi_0 = id_{\mathcal{A}};$$

this family is (i) a group if $I_0 = \mathbb{R}$, (ii) a semigroup if $I_0 = [0, \infty)$ and (iii) a local group if I_0 is an open interval containing the origin. (In the first two cases, the condition $s+t \in I_0$ above is obviously redundant.) One can define a linear, even mapping $g : D(g) \rightarrow \mathcal{A}$ where the domain $D(g)$ of g is a subset of \mathcal{A} such that, for all $A \in D(g)$, the limit

$$g(A) = \lim_{t \rightarrow 0} t^{-1}[\Phi_t(A) - A]$$

exists. This domain is clearly non-empty (the zero element lies in it). In general, $D(g)$ need not be dense in \mathcal{A} . Following (Yosida, [75]), $D(g)$ can be shown to be dense for the case $I_0 = [0, \infty)$ assuming sequential completeness for \mathcal{A} (which is a weaker condition than completeness which we have already assumed) and, for the family $\{\Phi_t\}$, the (C_0) condition

$$\lim_{t \rightarrow t_0} \Phi_t(A) = \Phi_{t_0}(A) \text{ for all } t_0 \in [0, \infty)$$

and the condition of equicontinuity in t [for a general treatment of equicontinuous mappings, see (Treves [70])]. We shall, however, generally employ a subclass of such one-parameter families (with \mathcal{A} a symplectic superalgebra; see section III A) for which the symplectic structure on \mathcal{A} ensures that the mapping g is defined on the whole of \mathcal{A} [see Eq.(35) below].

For small t , we have $\Phi_t(A) \simeq A + tg(A)$. The condition $\Phi_t(AB) = \Phi_t(A)\Phi_t(B)$ gives $g(AB) = g(A)B + Ag(B)$ implying that $g(A) = Y(A)$ for some even superderivation Y of \mathcal{A} (to be called the *infinitesimal generator* of Φ_t). From Eq.(3), we have, for small t ,

$$(\Phi_t)_* X \simeq X + t[Y, X] = X + tL_Y X. \quad (16)$$

Proposition 2.8. *Given Φ_t and Y as above and a p -form ω , we have, for small t ,*

$$\Phi_t^* \omega \simeq \omega - tL_Y \omega. \quad (17)$$

Proof. We have

$$\begin{aligned} (\Phi_t^* \omega)(X_1, \dots, X_p) &= \Phi_t^{-1}[\omega((\Phi_t)_* X_1, \dots, (\Phi_t)_* X_p)] \\ &\simeq \omega(X_1, \dots, X_p) - tY\omega(X_1, \dots, X_p) \\ &\quad + t \sum_{i=1}^p \omega(X_1, \dots, [Y, X_i], \dots, X_p) \\ &= [\omega - tL_Y \omega](X_1, \dots, X_p). \quad \square \end{aligned}$$

D. A generalization of the DVNCG scheme

It is easily seen that, in the developments in the last two subsections, it is possible to restrict the superderivations to a Lie sub-superalgebra \mathcal{X} of $SDer(\mathcal{A})$ and develop the whole formalism with the pair $(\mathcal{A}, \mathcal{X})$ obtaining thereby a useful generalization of the superderivation-based differential calculus. Working with such a pair is the analogue of restricting oneself to a leaf of a foliated manifold as the example below indicates.

Example. $\mathcal{A} = C^\infty(\mathbb{R}^3)$; \mathcal{X} = the Lie subalgebra of the Lie algebra $\mathcal{X}(\mathbb{R}^3)$ of vector fields on \mathbb{R}^3 generated by the Lie differential operators $L_j = \epsilon_{jkl} x_k \partial_l$ for the $SO(3)$ -action on \mathbb{R}^3 (here $\partial_l \equiv \frac{\partial}{\partial x^l}$). These differential operators, when expressed in terms of the polar coordinates r, θ, ϕ , contain only the partial derivatives with respect to θ and ϕ ; they, therefore, act on the 2-dimensional spheres that constitute the leaves of the foliation $\mathbb{R}^3 - \{(0, 0, 0)\} \simeq S^2 \times \mathbb{R}$. The restriction [of the pair $(\mathcal{A}, \mathcal{X}(\mathbb{R}^3))$] to $(\mathcal{A}, \mathcal{X})$ amounts to restricting oneself to a leaf (S^2) in the present case.

In the generalized formalism, one obtains the cochains $C^{p,s}(\mathcal{X}, \mathcal{A})$ for which the formulas of sections II B and II C are valid (with the X_j s restricted to \mathcal{X}). The differential forms $\Omega^{p,s}(\mathcal{A})$ will now be replaced by the objects $\Omega^{p,s}(\mathcal{X}, \mathcal{A})$ obtained by restricting the cochains to the $Z(\mathcal{A})$ -linear ones. [In the new notation, the objects $\Omega^{p,s}(\mathcal{A})$ is $\Omega^{p,s}(SDer(\mathcal{A}), \mathcal{A})$.]

To define the induced mappings Φ_* and Φ^* in the present context, one must employ a *pair-isomorphism* $\Phi : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{B}, \mathcal{Y})$ which consists of a

superalgebra \ast - isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that the induced mapping $\Phi_* : SDer(\mathcal{A}) \rightarrow SDer(\mathcal{B})$ restricts to an isomorphism of \mathcal{X} onto \mathcal{Y} . The properties of the induced mappings described in propositions 2.4 and 2.6 continue to hold.

Given a one-parameter family of transformations (i.e. pair isomorphisms) $\Phi_t : (\mathcal{A}, \mathcal{X}) \rightarrow (\mathcal{A}, \mathcal{X})$, the condition $(\Phi_t)_* \mathcal{X} \subset \mathcal{X}$ implies that the infinitesimal generator Y of Φ_t must satisfy the condition $[Y, X] \in \mathcal{X}$ for all $X \in \mathcal{X}$. In practical applications one will generally have $Y \in \mathcal{X}$ which trivially satisfies this condition.

This generalization will be used in sections III G and III H and in later papers.

E. Superderivations and differential forms on tensor products of superalgebras

Given two superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, their algebraic (skew) tensor product $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ has, as elements, finite sums of tensored pairs :

$$\sum_{j=1}^m A_j \otimes B_j \quad A_j \in \mathcal{A}^{(1)}, \quad B_j \in \mathcal{A}^{(2)}$$

with the multiplication rule (assuming B_j and A_k are homogeneous)

$$\left(\sum_{j=1}^m A_j \otimes B_j \right) \left(\sum_{k=1}^n A_k \otimes B_k \right) = \sum_{j,k} \eta_{B_j A_k} (A_j A_k) \otimes (B_j B_k).$$

Whenever topological considerations are relevant, the superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ will be understood to have the topology mentioned in section II A and their tensor product the projective topology [49].

The superalgebra $\mathcal{A}^{(1)}$ (resp. $\mathcal{A}^{(2)}$) has, in \mathcal{A} , an isomorphic copy consisting of the elements $(A \otimes I_2, A \in \mathcal{A}^{(1)})$ (resp. $(I_1 \otimes B, B \in \mathcal{A}^{(2)})$) to be denoted as $\tilde{\mathcal{A}}^{(1)}$ (resp. $\tilde{\mathcal{A}}^{(2)}$) where I_1 and I_2 are the unit elements of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$. We shall also use the notations $\tilde{A}^{(1)} = A \otimes I_2$ and $\tilde{B}^{(2)} = I_1 \otimes B$.

Superderivations and differential forms on $\mathcal{A}^{(i)}$ and $\tilde{\mathcal{A}}^{(i)}$ ($i = 1, 2$) are formally related through the induced mappings corresponding to the isomorphisms $\Xi^{(i)} : \mathcal{A}^{(i)} \rightarrow \tilde{\mathcal{A}}^{(i)}$ given by $\Xi^{(1)}(A) = A \otimes I_2$ and $\Xi^{(2)}(B) = I_1 \otimes B$. For example, corresponding to $X \in SDer(\mathcal{A}^{(1)})$, we have $\tilde{X}^{(1)} = \Xi_*^{(1)}(X)$ in $SDer(\tilde{\mathcal{A}}^{(1)})$ given by [see Eq.(3)]

$$\tilde{X}^{(1)}(\tilde{A}^{(1)}) = \Xi_*^{(1)}(X)(\tilde{A}^{(1)}) = \Xi^{(1)}[X(A)] = X(A) \otimes I_2. \quad (18)$$

Similarly, corresponding to $Y \in SDer(\mathcal{A}^{(2)})$, we have $\tilde{Y}^{(2)} \in SDer(\tilde{\mathcal{A}}^{(2)})$ given by $\tilde{Y}^{(2)}(\tilde{B}^2) = I_1 \otimes Y(B)$. For the 1-forms $\alpha \in \Omega^1(\mathcal{A}^{(1)})$ and $\beta \in \Omega^1(\mathcal{A}^{(2)})$, we have $\tilde{\alpha}^{(1)} \in \Omega^1(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{\beta}^{(2)} \in \Omega^1(\tilde{\mathcal{A}}^{(2)})$ given by [see Eq.(12)]

$$\tilde{\alpha}^{(1)}(\tilde{X}^{(1)}) = \Xi^{(1)}[\alpha(((\Xi^{(1)})^{-1})_* \tilde{X}^{(1)})] = \Xi^{(1)}[\alpha(X)] = \alpha(X) \otimes I_2 \quad (19)$$

and $\tilde{\beta}^{(2)}(\tilde{Y}^{(2)}) = I_1 \otimes \beta(Y)$. Analogous formulas hold for the higher forms.

We can extend the action of the superderivations $\tilde{X}^{(1)} \in SDer(\tilde{\mathcal{A}}^{(1)})$ and $\tilde{Y}^{(2)} \in SDer(\tilde{\mathcal{A}}^{(2)})$ to $\tilde{\mathcal{A}}^{(2)}$ and $\tilde{\mathcal{A}}^{(1)}$ respectively by defining

$$\tilde{X}^{(1)}(\tilde{B}^{(2)}) = 0, \quad \tilde{Y}^{(2)}(\tilde{A}^{(1)}) = 0 \quad \text{for all } A \in \mathcal{A}^{(1)} \text{ and } B \in \mathcal{A}^{(2)}.$$

It is useful to note that

$$A \otimes B = (A \otimes I_2)(I_1 \otimes B) = \tilde{A}^{(1)} \tilde{B}^{(2)};$$

hence, for homogeneous X and A,

$$X(A \otimes B) = X(\tilde{A}^{(1)} \tilde{B}^{(2)}) = (X \tilde{A}^{(1)}) \tilde{B}^{(2)} + \eta_{XA} \tilde{A}^{(1)} X(\tilde{B}^{(2)}). \quad (20)$$

It follows that an $X \in SDer(\mathcal{A})$ is determined completely by its action on the subalgebras $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$.

Proposition 2.9. *Given the superalgebras $\mathcal{A}^{(i)}, \tilde{\mathcal{A}}^{(i)}$ ($i = 1, 2$) and \mathcal{A} as above, every $X \in SDer(\mathcal{A})$ can be uniquely expressed as $X_1 + X_2$ where $X_i \in SDer(\mathcal{A})$ ($i = 1, 2$) satisfy the conditions*

- (i) $X_1 = X$ on $\tilde{\mathcal{A}}^{(1)}$ and $= 0$ on $\tilde{\mathcal{A}}^{(2)}$;
- (ii) $X_2 = X$ on $\tilde{\mathcal{A}}^{(2)}$ and $= 0$ on $\tilde{\mathcal{A}}^{(1)}$.

Proof. Clearly, it is adequate to consider homogeneous superderivations. Since X, as a mapping on \mathcal{A} , is linear, it will give, when acting on a general element of \mathcal{A} , a sum of terms of the form of the right hand side of Eq.(20). The reader can now define X_1 and X_2 in the obvious manner and complete the proof. \square

With the extensions described above, we have available to us superderivations belonging to the span of terms of the form [see Eq.(18)]

$$X = X^{(1)} \otimes I_2 + I_1 \otimes X^{(2)}. \quad (21)$$

Here I_1 and I_2 are to be understood as the linear mappings $\mu_1(I_1) = id_{\mathcal{A}^{(1)}}$ and $\mu_2(I_2) = id_{\mathcal{A}^{(2)}}$ where μ_i is the multiplication operator for the superalgebra

$\mathcal{A}^{(i)}$ ($i=1,2$). Replacing I_2 and I_1 in Eq.(21) by elements of $Z(\mathcal{A}^{(2)})$ and $Z(\mathcal{A}^{(1)})$ respectively, we again obtain superderivations of \mathcal{A} . We, therefore, have the space of superderivations

$$[SDer(\mathcal{A}^{(1)}) \otimes Z(\mathcal{A}^{(2)})] \oplus [Z(\mathcal{A}^{(1)}) \otimes SDer(\mathcal{A}^{(2)})]. \quad (22)$$

This space, however, is generally only a Lie sub-superalgebra of $SDer(\mathcal{A})$. For example, for $\mathcal{A}^{(1)} = M_m(\mathbb{C})$ and $\mathcal{A}^{(2)} = M_n(\mathbb{C})$ ($m, n > 1$), recalling that all the derivations of these matrix algebras are inner and that their centers consist of scalar multiples of the respective unit matrices, we have the (complex) dimensions of $Der(\mathcal{A}^{(1)})$, and $Der(\mathcal{A}^{(2)})$ respectively, $(m^2 - 1)$ and $(n^2 - 1)$ [so that the dimension of the space (22) is $m^2 + n^2 - 2$] whereas that of $Der(\mathcal{A})$ is $(m^2 n^2 - 1)$.

We shall need to employ (in section IV) a class of superderivations more general than (22). To introduce that class, it is instructive to obtain explicit representation(s) for a general derivation of the matrix algebra $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$. We have

$$[A \otimes B, C \otimes D]_{ir,js} = A_{ik} B_{rt} C_{kj} D_{ts} - C_{ik} D_{rt} A_{kj} B_{ts}$$

which gives

$$[A \otimes B, C \otimes D]_- = AC \otimes BD - CA \otimes DB \quad (23)$$

$$= [A, C]_- \otimes \frac{1}{2}[B, D]_+ + \frac{1}{2}[A, C]_+ \otimes [B, D]_-. \quad (24)$$

This gives, in obvious notation,

$$D_{A \otimes B} \equiv [A \otimes B, \cdot]_- = A.(.) \otimes B.(.) - (.).A \otimes (.).B \quad (25)$$

$$= D_A \otimes J_B^{(2)} + J_A^{(1)} \otimes D_B \quad (26)$$

where $J_B^{(2)}$ is the linear mapping on $\mathcal{A}^{(2)}$ given by $J_B^{(2)}(D) = \frac{1}{2}[B, D]_+$ and a similar expression for $J_A^{(1)}$ as a linear mapping on $\mathcal{A}^{(1)}$. Eq.(25) shows that a derivation of the algebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ need not explicitly contain those of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$. We shall, however, not get involved in the search for the most general expression for a (super-)derivation of the tensor product algebra \mathcal{A} (although such an expression would be very useful). The expression (26) is more useful for us; it is a special case of the more general form

$$X = X_1 \otimes \Psi_2 + \Psi_1 \otimes X_2 \quad (27)$$

where $X_i \in SDer(\mathcal{A}^{(i)})$ ($i=1,2$) and $\Psi_i : \mathcal{A}^{(i)} \rightarrow \mathcal{A}^{(i)}$ ($i=1,2$) are linear mappings. Our constructions in section IV A will lead us to structures of the form (27). It is important to note, however, that an expression of the form (27) (which represents a linear mapping of \mathcal{A} into itself) need not always be a (super-)derivation as can be easily checked. One should impose the condition (1) on such an expression to obtain a (super-)derivation.

A straightforward procedure to obtain general differential forms and the exterior derivative on \mathcal{A} is to obtain the graded differential algebra $(\Omega(\mathcal{A}), d)$ as the tensor product (Greub [43]) of the graded differential algebras $(\Omega(\mathcal{A}^{(1)}), d_1)$ and $(\Omega(\mathcal{A}^{(2)}), d_2)$. A differential form in $\Omega^{k,t}(\mathcal{A})$ is of the form [notation : $\alpha_{ir}^{(a)} \in \Omega^{i,r}(\mathcal{A}^{(a)})$, $a = 1,2$]

$$\alpha_{kt} = \sum_{\substack{i+j=k \\ r+s=t \bmod(2)}} \alpha_{ir}^{(1)} \otimes \alpha_{js}^{(2)}. \quad (*)$$

The d operation on $\Omega(\mathcal{A})$ is given by [here $\alpha \in \Omega^p(\mathcal{A}^{(1)})$ and $\beta \in \Omega(\mathcal{A}^{(2)})$]

$$d(\alpha \otimes \beta) = (d_1\alpha) \otimes \beta + (-1)^p \alpha \otimes d_2\beta. \quad (28)$$

We shall employ (in section III H and in later papers) exterior products of 1-forms on $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$. From the definition of exterior product in section II B, we have, for $\alpha, \beta \in \Omega^1(\mathcal{A})$ and homogeneous $X, Y \in SDer(\mathcal{A})$,

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \eta_{XY}\alpha(Y)\beta(X).$$

Superderivations of \mathcal{A} employed in concrete cases will be restricted to the form (27).

III. NONCOMMUTATIVE SYMPLECTIC GEOMETRY AND HAMILTONIAN MECHANICS

We shall now present a treatment of noncommutative symplectic geometry (extending the treatment of noncommutative symplectic structures by Dubois-Violette and coworkers mentioned above so as to include proper treatment of canonical transformations, Lie group actions etc in the noncommutative setting) and Hamiltonian mechanics along lines parallel to the developments in classical symplectic geometry and Hamiltonian mechanics (Arnold [4]; Woodhouse [74]; Abraham and Marsden [1]; Souriau [68]).

A. Symplectic structures; Poisson brackets

Note. The sign conventions adopted below are parallel to those of Woodhouse. This results in a (super-) Poisson bracket which, when applied to classical Hamiltonian mechanics, differs from the ‘usual’ one by a minus sign. [See Eq.(56).] The main virtue of the adopted conventions is that Eq.(33) below has no unpleasant minus sign on the right.

A *symplectic structure* on a superalgebra \mathcal{A} is a 2- form ω (the *symplectic form*) which is even, closed and *non-degenerate* in the sense that, for every $A \in \mathcal{A}$, there exists a unique superderivation Y_A in $SDer(\mathcal{A})$ such that

$$i_{Y_A}\omega = -dA. \quad (29)$$

The pair (\mathcal{A}, ω) will be called a *symplectic superalgebra*. A symplectic structure is said to be *exact* if the symplectic form is exact ($\omega = d\theta$ for some 1-form θ called the *symplectic potential*).

A *symplectic mapping* from a symplectic superalgebra (\mathcal{A}, α) to another one (\mathcal{B}, β) is a superalgebra isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\Phi^*\beta = \alpha$. [If the symplectic structures involved are exact, one requires a symplectic mapping to preserve the symplectic potential under the pull-back action; Eq.(14) then guarantees the preservation of the symplectic form.] A symplectic mapping from a symplectic superalgebra onto itself will be called a *canonical/symplectic transformation*. The symplectic form and its exterior powers are invariant under canonical transformations.

If Φ_t is a one-parameter family of canonical transformations generated by $X \in SDer(\mathcal{A})$, the condition $\Phi_t^*\omega = \omega$ implies, with Eq.(17),

$$L_X\omega = 0. \quad (30)$$

The argument just presented gives Eq.(30) with X an even superderivation. More generally, a superderivation X (even or odd or inhomogeneous) satisfying Eq.(30) will be called a *locally Hamiltonian* superderivation. Eq.(11) and the condition $d\omega = 0$ imply that Eq.(30) is equivalent to the condition

$$d(i_X\omega) = 0. \quad (31)$$

A (*globally*) *Hamiltonian superderivation* X is one for which the form $i_X\omega$ is exact. An example is the object Y_A defined by Eq.(29); it is called the Hamiltonian superderivation corresponding to $A \in \mathcal{A}$. Note from Eq.(29) that $\epsilon(Y_A) = \epsilon(A)$. In analogy with the commutative case, we have

Proposition 3.1. *The supercommutator of two locally Hamiltonian superderivations is a globally Hamiltonian superderivation.*

Proof. Given two locally Hamiltonian superderivations X and Y , we have

$$\begin{aligned}\eta_{X\omega} i_{[X,Y]}\omega &= (L_Y \circ i_X - i_X \circ L_Y)\omega \\ &= \eta_{Y\omega} (i_Y \circ d + d \circ i_Y)(i_X\omega) = \eta_{Y\omega} d(i_Y i_X\omega)\end{aligned}$$

which is exact. \square

It follows that the locally Hamiltonian superderivations constitute a Lie superalgebra in which the globally Hamiltonian superderivations constitute an ideal.

The *Poisson bracket* (PB) of two elements A and B of \mathcal{A} is defined as

$$\{A, B\} = \omega(Y_A, Y_B) = Y_A(B). \quad (32)$$

With A, B homogeneous, we have the super-analogue of the antisymmetry condition : $\{A, B\} = -\eta_{AB}\{B, A\}$ and of the Leibnitz rule :

$$\begin{aligned}\{A, BC\} = Y_A(BC) &= Y_A(B)C + \eta_{AB}BY_A(C) \\ &= \{A, B\}C + \eta_{AB}B\{A, C\}.\end{aligned}$$

As in the classical case, we have the relation

$$[Y_A, Y_B] = Y_{\{A, B\}}. \quad (33)$$

Eqn.(33) follows by using the equation for $i_{[X,Y]}\omega$ above with $X = Y_A$ and $Y = Y_B$ and equations (32) and (29), remembering that Eq.(29) determines Y_A uniquely. The super-Jacobi identity

$$\begin{aligned}0 &= \frac{1}{2}(d\omega)(Y_A, Y_B, Y_C) \\ &= \{A, \{B, C\}\} + (-1)^{\epsilon_A(\epsilon_B + \epsilon_C)}\{B, \{C, A\}\} \\ &\quad + (-1)^{\epsilon_C(\epsilon_A + \epsilon_B)}\{C, \{A, B\}\}\end{aligned} \quad (34)$$

is obtained by using Eq.(10) and noting that

$$\begin{aligned}Y_A[\omega(Y_B, Y_C)] &= \{A, \{B, C\}\} \\ \omega([Y_A, Y_B], Y_C) &= \omega(Y_{\{A, B\}}, Y_C) = \{\{A, B\}, C\}.\end{aligned}$$

Clearly, the pair $(\mathcal{A}, \{ , \})$ is a Lie superalgebra. Eq.(33) shows that the linear mapping $A \mapsto Y_A$ is a Lie-superalgebra homomorphism.

An element A of \mathcal{A} can act, via Y_A , as the infinitesimal generator of a one-parameter family of canonical transformations. The change in $B \in \mathcal{A}$ due to such an infinitesimal transformation is

$$\delta B = \epsilon Y_A(B) = \epsilon \{A, B\}. \quad (35)$$

In particular, if $\delta B = \epsilon I$ (infinitesimal ‘translation’ in B), we have

$$\{A, B\} = I \quad (36)$$

which is the noncommutative analogue of the classical PB relation $\{p, q\} = 1$. A pair (A, B) of elements of \mathcal{A} satisfying the condition (36) will be called a *canonical pair*.

B. Reality properties of the symplectic form and the Poisson bracket

For classical superdynamical systems, conventions about reality properties of the symplectic form are based on the fact that the Lagrangian is a real, even object (Berezin and Marinov [7]; Dass [22]). The matrix of the symplectic form is then real-antisymmetric in the ‘bosonic sector’ and imaginary-symmetric in the ‘fermionic sector’ (which means anti-Hermitian in both sectors). Keeping this in view, it appears appropriate to impose, in noncommutative Hamiltonian mechanics, the following reality condition on the symplectic form ω :

$$\omega^*(X, Y) = -\eta_{XY}\omega(Y, X) \quad \text{for all homogeneous } X, Y \in SDer(\mathcal{A}); \quad (37)$$

but this means, by Eq.(7), that $\omega^* = \omega$ (i.e. ω is real) which is the most natural assumption to make about ω . Eq.(37) is equivalent to the condition

$$\omega(X^*, Y^*) = -\eta_{XY}[\omega(Y, X)]^*.$$

Proposition 3.2. *In a symplectic superalgebra (\mathcal{A}, ω) with ω real, we have, for arbitrary $A, B \in \mathcal{A}$,*

$$(i) (Y_A)^* = Y_{A^*}; \quad (ii) \{A, B\}^* = \{A^*, B^*\}. \quad (38)$$

Proof. (i) Eq.(29) gives, for any $X \in SDer(\mathcal{A})$,

$$-(dA)(X) = \omega(Y_A, X) = [\omega(Y_A^*, X^*)]^*,$$

which, on applying the $*$ -operation, gives

$$(i_{Y_A^*}\omega)(X^*) = -[dA(X)]^* = -dA^*(X^*)$$

which, in turn, gives $i_{Y_A^*}\omega = -dA^*$ implying the desired result.

(ii) We have

$$\begin{aligned}\{A, B\}^* &= [\omega(Y_A, Y_B)]^* = \omega^*(Y_A^*, Y_B^*) \\ &= \omega(Y_{A^*}, Y_{B^*}) = \{A^*, B^*\}. \quad \square\end{aligned}$$

Eq.(38)(ii) is consistent with the reality properties of the classical and quantum Poisson brackets.[See equations (56) and (43) below.] Note that the PB of two hermitian elements is hermitian.

C. Special algebras; the canonical symplectic form

In this subsection, we shall consider a distinguished class of superalgebras which have a canonical symplectic structure associated with them. As we shall see below and in paper II, these superalgebras play an important role in quantum mechanics.

A complex, associative, non-supercommutative superalgebra will be called *special* if all its superderivations are inner. The differential 2-form ω_c defined on such a superalgebra \mathcal{A} by

$$\omega_c(D_A, D_B) = [A, B] \tag{39}$$

is said to be the *canonical form* on \mathcal{A} . (*Note.* This definition of the canonical form differs from that of Dubois-Violette [31] by a factor of i .) It is easily seen to be closed [the equation $(d\omega_c)(D_A, D_B, D_C) = 0$ is nothing but the Jacobi identity for the supercommutator], imaginary (i.e. $\omega_c^* = -\omega_c$) and dimensionless. For any $A \in \mathcal{A}$, the equation

$$\omega_c(Y_A, D_B) = -(dA)(D_B) = [A, B] \text{ for all } B \in \mathcal{A}$$

has the unique solution $Y_A = D_A$. (To see this, note that, since all superderivations are inner, $Y_A = D_C$ for some $C \in \mathcal{A}$; the condition $\omega_c(D_C, D_B) = [C, B] = [A, B]$ for all $B \in \mathcal{A}$ implies that $(C - A) \in Z(\mathcal{A})$. But then $D_C = D_A$. \square) This shows that ω_c is nondegenerate and we have

$$i_{D_A}\omega_c = -dA. \tag{40}$$

The closed and non-degenerate form ω_c defines, on \mathcal{A} , the *canonical symplectic structure*. It gives, as Poisson bracket, the supercommutator :

$$\{A, B\} = Y_A(B) = D_A(B) = [A, B]. \quad (41)$$

Using equations (40) and (11), it is easily seen that the form ω_c is *invariant* in the sense that $L_X \omega_c = 0$ for all $X \in SDer(\mathcal{A})$. The invariant symplectic structure on the algebra $M_n(C)$ of complex $n \times n$ matrices obtained by Dubois-Violette et al [33] is a special case of the invariant canonical symplectic structure on special superalgebras described above.

If, for a special superalgebra \mathcal{A} , we take, instead of ω_c , $\omega = b \omega_c$ as the symplectic form (where b is a nonzero complex number), we have

$$Y_A = b^{-1} D_A, \quad \{A, B\} = b^{-1} [A, B]. \quad (42)$$

The *quantum Poisson bracket*

$$\{A, B\}_Q = (-i\hbar)^{-1} [A, B] \quad (43)$$

is a special case of this with $b = -i\hbar$. (Note that b must be imaginary to make ω real.) In the case of the Schrödinger representation for a nonrelativistic spinless particle, the Heisenberg-Schrödinger algebra \mathcal{A}_Q generated by the position and momentum operators X_j, P_j ($j=1,2,3$) defined on an invariant dense domain in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx)$ is special [31,32]; one has, therefore, a canonical form ω_c and the *quantum symplectic structure* on \mathcal{A}_Q given by the *quantum symplectic form*

$$\omega_Q = -i\hbar \omega_c \quad (44)$$

which gives the PB (43).

We shall refer to a symplectic structure of the above sort with a general nonzero b as the *quantum symplectic structure with parameter b* .

D. Noncommutative Hamiltonian mechanics

We shall now present the formalism of NHM combining elements of noncommutative symplectic geometry and noncommutative probability in the algebraic framework developed above.

The system algebra and states

In NHM, one associates, with every physical system, a symplectic super-algebra (\mathcal{A}, ω) . Here we shall treat the term ‘physical system’ informally as is traditionally done; some formalities in this connection will be taken care of in paper II where a provisional set of axioms will be given. The even Hermitian elements of \mathcal{A} will be identified as *observables* of the system. The collection of all observables in \mathcal{A} will be denoted as $\mathcal{O}(\mathcal{A})$; it is a real linear space closed under PBs and, therefore, constitutes a real Lie subalgebra of the complex Lie super-algebra $(\mathcal{A}, \{, \})$.

To take care of limit processes and continuity of mappings, we must employ topological algebras. The choice of the admissible class of topological algebras must meet the following reasonable requirements:

- (i) It should be closed under the formation of (a) topological completions and (b) tensor products. (Both are nontrivial requirements [29].)
- (ii) It should include
 - (a) the Op^* -algebras (Horuzhy [51]) based on the pairs $(\mathcal{H}, \mathcal{D})$ where \mathcal{H} is a complex separable Hilbert space and \mathcal{D} a dense linear subset of \mathcal{H} ; [Such an algebra is an algebra of operators which, along with their adjoints, map \mathcal{D} into itself. The $*$ -operation on the algebra is defined as the restriction of the Hilbert space adjoint on \mathcal{D} . These are the algebras of operators (not necessarily bounded) appearing in the traditional Hilbert space QM; for example, the operator algebra \mathcal{A}_Q in the previous subsection belongs to this class.]
 - (b) algebras of smooth functions on manifolds (to accommodate classical Hamiltonian mechanics and permit a transparent treatment of quantum-classical correspondence).
- (iii) The GNS representations of the system algebra induced by various states must have *separable* Hilbert spaces as the representation spaces.

The right choice appears to be, as mentioned in section II A, the $\hat{\otimes}$ -(star-) algebras of Helemskii [49] (i.e. locally convex $*$ -algebras which are complete and Hausdorff with a jointly continuous product) satisfying the additional condition of being separable. Henceforth (super-)algebras employed in the treatment of NHM will generally be assumed to belong to this class. Any additional requirements will be explicitly stated if and when needed in later work.

A *state* on a system algebra \mathcal{A} is a continuous linear functional ϕ on \mathcal{A} which is (i) positive [which means $\phi(A^*A) \geq 0$ for all $A \in \mathcal{A}$] and (ii) normalized [i.e. $\phi(I) = 1$]. Given a state ϕ , the quantity $\phi(A)$ for any observable A is real (this can be seen by considering, for example, the

quantity $\phi[(I + A)^*(I + A)]$ and is to be interpreted as the expectation value of A in the state ϕ . Following general usage in literature, we shall call observables of the form A^*A or a sum of such terms *positive* (strictly speaking, the term ‘non-negative’ would be more appropriate); states assign non-negative expectation values to such observables. The family of all states on \mathcal{A} will be denoted as $\mathcal{S}(\mathcal{A})$. It is closed under convex combinations: given $\phi_i \in \mathcal{S}(\mathcal{A})$, $i = 1, \dots, n$ and $p_i > 0$ with $p_1 + \dots + p_n = 1$, we have $\phi = \sum_{i=1}^n p_i \phi_i$ also in $\mathcal{S}(\mathcal{A})$. States which cannot be expressed as nontrivial convex combinations of other states will be called *pure* states and others *mixed* states. The family of pure states of \mathcal{A} will be denoted as $\mathcal{S}_1(\mathcal{A})$. The triple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega)$ will be referred to as a *symplectic triple*. This is the proper noncommutative analogue of a symplectic manifold (M, ω_{cl}) in classical Hamiltonian mechanics. [Note that specification of the phase space M serves to define both the algebra $C^\infty(M)$ as well as its pure states which are points of M .]

Expectation values of all even elements of \mathcal{A} can be expressed in terms of those of the observables (by considering the breakup of such an element into its Hermitian and anti-Hermitian part). This leaves out the odd elements of \mathcal{A} . It appears reasonable to demand that the expectation values $\phi(A)$ of all odd elements $A \in \mathcal{A}$ must vanish for all pure states (and, therefore, for all states).

Denoting the dual of the superalgebra \mathcal{A} by \mathcal{A}^* , an automorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ induces the transpose mapping $\tilde{\Phi} : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \text{ or } \langle \tilde{\Phi}(\phi), A \rangle = \langle \phi, \Phi(A) \rangle \quad (45)$$

where the second alternative has employed the dual space pairing \langle, \rangle . The mapping $\tilde{\Phi}$ (which is easily seen to be linear and bijective) maps states (which form a subset of \mathcal{A}^*) onto states. To see this, note that

$$(i) \quad \tilde{\Phi}(\phi)(A^*A) = \phi(\Phi(A^*A)) = \phi(\Phi(A)^*\Phi(A)) \geq 0;$$

$$(ii) \quad [\tilde{\Phi}(\phi)](I) = \phi(\Phi(I)) = \phi(I) = 1.$$

The linearity of $\tilde{\Phi}$ (as a mapping on \mathcal{A}^*) ensures that it preserves convex combinations of states. In particular, it maps pure states onto pure states. We have, therefore, a bijective mapping $\tilde{\Phi} : \mathcal{S}_1(\mathcal{A}) \rightarrow \mathcal{S}_1(\mathcal{A})$. An automorphism of the set $\mathcal{S}(\mathcal{A})$ of states is an invertible mapping $\Psi : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A})$ preserving convex combinations. The mapping $\tilde{\Phi}$ is, therefore, an automorphism of the set $\mathcal{S}(\mathcal{A})$ of states restricting to a bijection on the set $\mathcal{S}_1(\mathcal{A})$ of pure states.

When Φ is a canonical transformation, the condition $\Phi^*\omega = \omega$ gives, for $X, Y \in SDer(\mathcal{A})$,

$$\omega(X, Y) = (\Phi^*\omega)(X, Y) = \Phi^{-1}[\omega(\Phi_*X, \Phi_*Y)]$$

which gives

$$\Phi[\omega(X, Y)] = \omega(\Phi_*X, \Phi_*Y). \quad (46)$$

Taking expectation value of both sides of this equation in a state ϕ , we get

$$(\tilde{\Phi}\phi)[\omega(X, Y)] = \phi[\omega(\Phi_*X, \Phi_*Y)]. \quad (47)$$

When Φ is an infinitesimal canonical transformation generated by $G \in \mathcal{A}$, we have

$$\tilde{\Phi}(\phi)(A) = \phi(\Phi(A)) \simeq \phi(A + \epsilon\{G, A\}).$$

Putting $\tilde{\Phi}(\phi) = \phi + \delta\phi$, we have

$$(\delta\phi)(A) = \epsilon\phi(\{G, A\}). \quad (48)$$

Dynamics

Dynamics of the system is described in terms of the one-parameter family Φ_t of canonical transformations generated by an observable H , called the *Hamiltonian*. (The parameter t is supposed to be an evolution parameter which need not always be the conventional time.) Writing $\Phi_t(A) = A(t)$ and recalling Eq.(35), we have the *Hamilton's equation* of NHM :

$$\frac{dA(t)}{dt} = Y_H[A(t)] = \{H, A(t)\}. \quad (49)$$

The triple (\mathcal{A}, ω, H) [or, more appropriately, the quadruple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$] will be called an *NHM Hamiltonian system* or simply a *noncommutative Hamiltonian system*. It is the analogue of a classical Hamiltonian system (M, ω_{cl}, H_{cl}) [where (M, ω_{cl}) is a symplectic manifold and H_{cl} , the classical Hamiltonian (a smooth real-valued function on M)]. As far as the evolution is concerned, the Hamiltonian is, as in the classical case, arbitrary up to the addition of a constant multiple of the unit element. We shall assume that H

is bounded below in the sense that its expectation values in all pure states (hence in all states) are bounded below.

This is the analogue of the Heisenberg picture in traditional QM. An equivalent description, the analogue of the Schrödinger picture, is obtained by transferring the time dependence to states through the relation [see Eq.(45)]

$$\langle \phi(t), A \rangle = \langle \phi, A(t) \rangle \quad (50)$$

where $\phi(t) = \tilde{\Phi}_t(\phi)$. The mapping $\tilde{\Phi}_t$ satisfies the condition (47) [with $\Phi = \Phi_t$] which may be said to represent the canonicity of the evolution of states. With $\Phi = \Phi_t$ and $G = H$, Eq.(48) gives the *Liouville equation* of NHM:

$$\frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \quad \text{or} \quad \frac{d\phi(t)}{dt}(\cdot) = \phi(t)(\{H, \cdot\}). \quad (51)$$

Equivalent descriptions; Symmetries and conservation laws

By a ‘description’ of a system, we shall mean specification of its triple $(\mathcal{A}, \mathcal{S}(\mathcal{A}), \omega)$. Two descriptions are said to be *equivalent* if they are related through a pair of automorphisms $\Phi_1 : \mathcal{A} \rightarrow \mathcal{A}$ and $\Phi_2 : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A})$ such that the symplectic form and the expectation values are preserved :

$$\Phi_1^* \omega = \omega; \quad \Phi_2(\phi)[\Phi_1(A)] = \phi(A) \quad (52)$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$. The second equation above and Eq(45) imply that we must have $\Phi_2 = (\tilde{\Phi}_1)^{-1}$. Two equivalent descriptions are, therefore, related through a canonical transformation on \mathcal{A} and the corresponding inverse transpose transformation on the states. An infinitesimal transformation of this type generated by $G \in \mathcal{A}$ takes the form [see equations (35) and (48)]

$$\delta A = \epsilon \{G, A\}, \quad (\delta \phi)(A) = -\epsilon \phi(\{G, A\}) \quad (53)$$

for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}(\mathcal{A})$.

These transformations may be called symmetries of the formalism; they are the analogues of simultaneous unitary transformations on operators and state vectors in a Hilbert space preserving expectation values of operators. Symmetries of dynamics are the subclass of these which leave the Hamiltonian invariant:

$$\Phi_1(H) = H. \quad (54)$$

For an infinitesimal transformation generated by $G \in \mathcal{A}$, this equation gives

$$\{G, H\} = 0. \quad (55)$$

It now follows from the Hamilton's equation (49) that (in the 'Heisenberg picture' evolution) G is a constant of motion. This is the situation familiar from classical and quantum mechanics: generators of symmetries of the Hamiltonian are conserved quantities and vice-versa.

Note. Redundancy in the definition of symmetry operations given above permits some flexibility in their implementation. It is often useful to implement them such that they act, in a single implementation, *either* on observables *or* on states, and the two actions are related as the Heisenberg and Schrödinger picture evolutions above [see equations (50) and (45)]; we shall refer to this type of implementation as *unimodal*. In such an implementation, the second equation in (53) will not have a minus sign on the right.

For future reference, we define equivalence of NHM Hamiltonian systems. Two NHM Hamiltonian systems $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ and $(\mathcal{A}', \mathcal{S}_1(\mathcal{A}'), \omega', H')$ are said to be equivalent if they are related through a pair $\Phi = (\Phi_1, \Phi_2)$ of bijective mappings such that $\Phi_1 : (\mathcal{A}, \omega) \rightarrow (\mathcal{A}', \omega')$ is a symplectic mapping connecting the Hamiltonians [i.e. $\Phi_1^* \omega' = \omega$ and $\Phi_1(H) = H'$] and $\Phi_2 : \mathcal{S}_1(\mathcal{A}) \rightarrow \mathcal{S}_1(\mathcal{A}')$ such that $\langle \Phi_2(\phi), \Phi_1(A) \rangle = \langle \phi, A \rangle$ for all $A \in \mathcal{A}$ and $\phi \in \mathcal{S}_1(\mathcal{A})$.

Classical Hamiltonian mechanics and traditional Hilbert space QM as subdisciplines of NHM

A classical hamiltonian system (M, ω_{cl}, H_{cl}) is a special case of an NHM Hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ with $\mathcal{A} = \mathcal{A}_{cl} \equiv C^\infty(M)$, $\mathcal{S}_1(\mathcal{A}) = M$, $\omega = \omega_{cl} \equiv \sum dp_j \wedge dq^j$ (in canonical coordinates) and $H = H_{cl}$; the NHM PBs are now the classical PBs

$$\{f, g\}_{cl} = \sum_{j=1}^n \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right). \quad (56)$$

Eq.(49) now becomes the classical Hamilton's equation. Representing states by probability densities in phase space, Eq.(51) goes over, in appropriate cases (for $M = R^{2n}$, for example, after the obvious partial integrations) to the classical Liouville equation for the density function.

To see the traditional Hilbert space QM as a subdiscipline of NHM, it is useful to introduce the concept of a *quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where \mathcal{H} is a complex separable Hilbert space, \mathcal{D} a dense linear subset of \mathcal{H} and \mathcal{A} an Op^* -algebra of operators based on $(\mathcal{H}, \mathcal{D})$. Here we shall consider only the *standard quantum triples* by which we mean those in which (i) the algebra \mathcal{A} is special in the sense of section III C, and (ii) \mathcal{A} acts irreducibly on $(\mathcal{H}, \mathcal{D})$ [i.e. there does not exist a smaller quantum triple $(\mathcal{H}', \mathcal{D}', \mathcal{A})$ with $\mathcal{D}' \subset \mathcal{D}$, $\mathcal{A}\mathcal{D}' \subset \mathcal{D}'$ and \mathcal{H}' is a proper subspace of \mathcal{H}]; more general situations involving superselection rules will be covered in the systematic treatment of quantum systems in paper II.

With \mathcal{A} special, one can define the quantum symplectic form ω_Q as in Eq.(44); the pair (\mathcal{A}, ω_Q) may be called a *quantum symplectic algebra*. With \mathcal{A} -action irreducible, the space $\mathcal{S}_1(\mathcal{A})$ of pure states of \mathcal{A} consists of vector states corresponding to normalized vectors in \mathcal{D} . Choosing an appropriate self adjoint element H of \mathcal{A} as the Hamiltonian operator, we have a quantum hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega_Q, H)$ as a special case of an NHM Hamiltonian system. With the quantum PBs of Eq.(43), Eq.(49) goes over to the traditional Heisenberg equation of motion. General states are represented by density operators ρ satisfying the condition $|Tr(\overline{\rho A})| < \infty$ for all $A \in \mathcal{A}$ where the overbar indicates closure ([29]; p. 121). Noting that $Tr(\overline{\rho_1 A}) = Tr(\overline{\rho_2 A})$ for all $A \in \mathcal{A}$ implies $\rho_1 = \rho_2$, Eq.(51) goes over to the von Neumann equation

$$\frac{d\rho(t)}{dt} = (-i\hbar)^{-1}[\rho(t), H]. \quad (57)$$

E. Symplectic actions of Lie groups

The study of symplectic actions of Lie groups in NHM proceeds generally parallel to the classical case (Sudarshan and Mukunda [69]; Alonso [2]; Arnold [4]; Guillemin and Sternberg [45]; Woodhouse [74]) and promises to be quite rich and rewarding. Here we shall present the essential developments mainly to provide background material for sections III F and III H and paper II.

Let G be a connected Lie group with Lie algebra \mathcal{G} . Elements of G , \mathcal{G} and \mathcal{G}^* (the dual space of \mathcal{G}) will be denoted, respectively, as g, h, \dots , ξ, η, \dots and λ, μ, \dots . The pairing between \mathcal{G}^* and \mathcal{G} will be denoted as $\langle \cdot, \cdot \rangle$. Choosing a basis $\{\xi_a; a = 1, \dots, r\}$ in \mathcal{G} , we have the commutation relations $[\xi_a, \xi_b] = C_{ab}^c \xi_c$. The dual basis in \mathcal{G}^* is denoted as $\{\lambda^a\}$ (so that $\langle \lambda^a, \xi_b \rangle = \delta_b^a$). The action

of G on \mathcal{G} (adjoint representation) will be denoted as $Ad_g : \mathcal{G} \rightarrow \mathcal{G}$ and that on \mathcal{G}^* (the coadjoint representation) by $Cad_g : \mathcal{G}^* \rightarrow \mathcal{G}^*$; the two are related as $\langle Cad_g \lambda, \xi \rangle = \langle \lambda, Ad_{g^{-1}} \xi \rangle$. With the bases chosen as above, the matrices in the two representations are related as $(Cad_g)_{ab} = (Ad_{g^{-1}})_{ba}$.

Recalling the mappings Φ_1 and Φ_2 of the previous subsection, a *symplectic action* of G on a symplectic superalgebra (\mathcal{A}, ω) is given by the assignment, to each $g \in G$, a symplectic mapping (canonical transformation) $\Phi_1(g) : \mathcal{A} \rightarrow \mathcal{A}$ which is a group action [which means that $\Phi_1(g)\Phi_1(h) = \Phi_1(gh)$ and $\Phi_1(e) = id_{\mathcal{A}}$ in obvious notation]. The action on the states is given by the mappings $\Phi_2(g) = [\tilde{\Phi}_1(g)]^{-1}$.

A one-parameter subgroup $g(t)$ of G generated by $\xi \in \mathcal{G}$ induces a locally Hamiltonian superderivation $Z_\xi \in SDer(\mathcal{A})$ as the generator of the one-parameter family $\Phi_1(g(t)^{-1})$ of canonical transformations of \mathcal{A} : For small t

$$\Phi_1(g(t)^{-1})(A) \simeq A + tZ_\xi(A).$$

[*Note.* We employed $\Phi_1(g(t)^{-1})$ (and not $\Phi_1(g(t))$) for defining Z_ξ above because the former correspond to a right action of G on \mathcal{A} .]

Proposition 3.3. *The correspondence $\xi \rightarrow Z_\xi$ is a Lie algebra homomorphism :*

$$Z_{[\xi, \eta]} = [Z_\xi, Z_\eta]. \quad (58)$$

A proof of (58), whose steps are parallel to those for Lie group actions on manifolds (Matsushima [57]; Dass [22]), was given in [24] [it is an instructive application of the mathematical techniques of section II; in particular, the induced mappings Φ_* of Eq.(3) play a role analogous to that of the mappings on vector fields induced by diffeomorphisms]; we shall skip the details here.

The G -action is said to be *hamiltonian* if the superderivations Z_ξ are Hamiltonian, i.e. for each $\xi \in \mathcal{G}$, $Z_\xi = Y_{h_\xi}$ for some $h_\xi \in \mathcal{A}$ (called the *hamiltonian* corresponding to ξ). These hamiltonians are arbitrary up to addition of multiples of the unit element. This arbitrariness can be somewhat reduced by insisting that h_ξ be linear in ξ . (This can be achieved by first defining the hamiltonians for the members of a basis in \mathcal{G} and then for general elements as appropriate linear combinations of these.) We shall always assume this linearity.

A hamiltonian G -action satisfying the additional condition

$$\{h_\xi, h_\eta\} = h_{[\xi, \eta]} \text{ for all } \xi, \eta \in \mathcal{G} \quad (59)$$

is called a *Poisson action*. Note that (recalling the statement after proposition 3.2) the hamiltonians of a Poisson action can be consistently chosen to be observables.

Proposition 3.4. *The hamiltonians of the Poisson action of a connected Lie group G on a symplectic superalgebra (\mathcal{A}, ω) have the following equivariance property:*

$$\Phi_1(g)(h_\xi) = h_{Ad_g(\xi)} \quad \forall g \in G \quad \text{and} \quad \xi \in \mathcal{G}. \quad (60)$$

Proof. Since G is connected, it is adequate to verify this relation for infinitesimal group actions. Denoting by $g(t)$ the one-parameter group generated by $\eta \in \mathcal{G}$, we have, for small t ,

$$\Phi_1(g(t))(h_\xi) \simeq h_\xi + t\{h_\eta, h_\xi\} = h_\xi + th_{[\eta, \xi]} = h_{\xi+t[\eta, \xi]} \simeq h_{Ad_{g(t)}\xi}$$

completing the verification. \square

A Poisson action is not always admissible. The obstruction to such an action is determined by the objects

$$\alpha(\xi, \eta) = \{h_\xi, h_\eta\} - h_{[\xi, \eta]} \quad (61)$$

which are easily seen to have vanishing Hamiltonian derivations :

$$Y_{\alpha(\xi, \eta)} = [Y_{h_\xi}, Y_{h_\eta}] - Y_{h_{[\xi, \eta]}} = [Z_\xi, Z_\eta] - Z_{[\xi, \eta]} = 0$$

and hence vanishing Poisson brackets with all elements of \mathcal{A} . [This last condition defines the so-called *neutral elements* [69] of the Lie algebra $(\mathcal{A}, \{, \})$. They clearly form a vector space which will be denoted as \mathcal{N} .] We also have

$$\alpha([\xi, \eta], \zeta) + \alpha([\eta, \zeta], \xi) + \alpha([\zeta, \xi], \eta) = 0 \quad \text{for all } \xi, \eta, \zeta \in \mathcal{G}.$$

The derivation (Woodhouse [74]; p.44) of this result in classical mechanics employs only the standard properties of PBs and remains valid in NHM. Comparing this equation with Eq.(6), we see that $\alpha(.,.) \in Z_0^2(\mathcal{G}, \mathcal{N})$. A redefinition of the hamiltonians $h_\xi \rightarrow h'_\xi = h_\xi + k_\xi I$ (where the scalars k_ξ are linear in ξ) changes α by a coboundary term:

$$\alpha'(\xi, \eta) = \alpha(\xi, \eta) - k_{[\xi, \eta]} I$$

showing that the obstruction is characterized by a cohomology class of \mathcal{G} [i.e. an element of $H_0^2(\mathcal{G}, \mathcal{N})$]. A necessary and sufficient condition for the

admissibility of Poisson action of G on \mathcal{A} is that it should be possible to transform away all the obstruction 2-cocycles by redefining the hamiltonians, or, equivalently, $H_0^2(\mathcal{G}, \mathcal{N}) = 0$.

We now restrict ourselves to the special case, relevant for application in paper II (in the treatment of elementary systems), in which the cocycles α are multiples of the unit element :

$$\alpha(\xi, \eta) = \underline{\alpha}(\xi, \eta)I. \quad (62)$$

Assuming that the hamiltonians h_ξ are observables, the quantities $\underline{\alpha}(\xi, \eta)$ must be real numbers. This implies $\mathcal{N} = \mathbb{R}$. In this case, the relevant cohomology group $H_0^2(\mathcal{G}, \mathbb{R})$ is a real finite dimensional vector space; we shall take it to be \mathbb{R}^m . In this case, as in classical Hamiltonian mechanics [69,13,2], hamiltonian group actions (more generally, Lie algebra actions) with nontrivial neutral elements can be treated as Poisson actions of a (Lie group with a) larger Lie algebra $\hat{\mathcal{G}}$ obtained as follows: Let $\eta_r(.,.)(r = 1, \dots, m)$ be a set of representatives in $Z_0^2(\mathcal{G}, \mathbb{R})$ of a basis in $H_0^2(\mathcal{G}, \mathbb{R})$. We add extra generators M_r ($r = 1, \dots, m$) to the basis $\{\xi_a\}$ of \mathcal{G} and take the commutation relations of the larger Lie algebra $\hat{\mathcal{G}}$ as

$$[\xi_a, \xi_b] = C_{ab}^c \xi_c + \sum_{r=1}^m \eta_r(\xi_a, \xi_b) M_r; \quad [\xi_a, M_r] = 0 = [M_r, M_s]. \quad (63)$$

The connected and simply connected Lie group \hat{G} with the Lie algebra $\hat{\mathcal{G}}$ is called the *projective group* of G (called ‘projective covering group’ of G by Cariñena and Santander [13]; we follow the terminology of Alonso [2]); it is generally a central extension of the universal covering group \tilde{G} of G .

The hamiltonian action of G with the cocycle α now becomes a Poisson action of \hat{G} with the Poisson bracket relations (writing $h_{\xi_a} = h_a$, $h_{M_r} = h_r$)

$$\{h_a, h_b\} = C_{ab}^c h_c + \sum_{r=1}^m \eta_r(\xi_a, \xi_b) h_r; \quad \{h_a, h_r\} = 0 = \{h_r, h_s\}. \quad (64)$$

F. The momentum map.

In classical mechanics, given a Poisson action of a connected Lie group G on a symplectic manifold (M, ω_{cl}) [with hamiltonians $h_\xi^{(cl)} \in C^\infty(M)$], a

useful construction is the so-called *momentum map* (Souriau [68]; Arnold [4]; Guillemin and Sternberg [45]) $P : M \rightarrow \mathcal{G}^*$ given by

$$\langle P(x), \xi \rangle = h_\xi^{(cl)}(x) \quad \forall x \in M \text{ and } \xi \in \mathcal{G}. \quad (65)$$

This map relates the symplectic action Φ_g of G on M ($\Phi_g : M \rightarrow M, \Phi_g^* \omega_{cl} = \omega_{cl} \forall g \in G$) and the transposed adjoint action on \mathcal{G}^* through the equivariance property

$$P(\Phi_g(x)) = \widetilde{Ad_g}(P(x)) \quad \forall x \in M \text{ and } g \in G. \quad (66)$$

Noting that points of M are pure states of the algebra $\mathcal{A}_{cl} = C^\infty(M)$, the map P may be considered as the restriction to M of the dual/transpose $\tilde{h}^{(cl)} : \mathcal{A}_{cl}^* \rightarrow \mathcal{G}^*$ (where \mathcal{A}_{cl}^* is the dual space of \mathcal{A}_{cl}) of the linear map $h^{(cl)} : \mathcal{G} \rightarrow \mathcal{A}_{cl}$ [given by $h^{(cl)}(\xi) = h_\xi^{(cl)}$]:

$$\langle \tilde{h}^{(cl)}(u), \xi \rangle = \langle u, h^{(cl)}(\xi) \rangle \quad \forall u \in \mathcal{A}_{cl}^* \text{ and } \xi \in \mathcal{G}.$$

The analogue of M in NHM is $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{A})$. Defining $h : \mathcal{G} \rightarrow \mathcal{A}$ by $h(\xi) = h_\xi$, the analogue of the momentum map in NHM is the mapping $\tilde{h} : \mathcal{S}_1 \rightarrow \mathcal{G}^*$ (considered as the restriction to \mathcal{S}_1 of the mapping $\tilde{h} : \mathcal{A}^* \rightarrow \mathcal{G}^*$) given by

$$\langle \tilde{h}(\phi), \xi \rangle = \langle \phi, h(\xi) \rangle = \langle \phi, h_\xi \rangle \quad \text{for all } \phi \in \mathcal{S}_1 \text{ and } \xi \in \mathcal{G}. \quad (67)$$

Proposition 3.5. *The noncommutative momentum map \tilde{h} has the following equivariance property: In the notation employed above*

$$\tilde{h}(\Phi_2(g)\phi) = \text{Cad}_g(\tilde{h}(\phi)) \quad \text{for all } \phi \in \mathcal{S}_1 \text{ and } g \in G. \quad (68)$$

Proof. We have, for any $\xi \in \mathcal{G}$,

$$\begin{aligned} \langle \tilde{h}(\Phi_2(g)\phi), \xi \rangle &= \langle \Phi_2(g)\phi, h_\xi \rangle = \langle \phi, \Phi_1(g^{-1})(h_\xi) \rangle = \langle \phi, h_{Ad_{g^{-1}}(\xi)} \rangle \\ &= \langle \phi, h(Ad_{g^{-1}}(\xi)) \rangle = \langle \text{Cad}_g(\tilde{h}(\phi)), \xi \rangle \end{aligned}$$

giving Eq.(68). \square

Note. In Eq.(68), the co-adjoint (and not the transposed adjoint) action appears on the right because $\Phi_2(g)$ is inverse transpose (and not transpose) of $\Phi_1(g)$. With this understanding, (66) is obviously a special case of (68).

We shall make use of the noncommutative momentum map in the treatment of elementary systems in paper II.

G. Generalized symplectic structures and Hamiltonian systems

The generalization of the DVNCG scheme introduced in section II D can be employed to obtain the corresponding generalization of the NHM formalism. One picks up a distinguished Lie sub-superalgebra \mathcal{X} of $SDer(\mathcal{A})$ and restricts the superderivations of \mathcal{A} in all definitions and constructions to those in \mathcal{X} . Thus, a symplectic superalgebra (\mathcal{A}, ω) is now replaced by a *generalized symplectic superalgebra* $(\mathcal{A}, \mathcal{X}, \omega)$ and a symplectic mappings $\Phi : (\mathcal{A}, \mathcal{X}, \alpha) \rightarrow (\mathcal{B}, \mathcal{Y}, \beta)$ is restricted to a superalgebra-isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\Phi_* : \mathcal{X} \rightarrow \mathcal{Y}$ is a Lie-superalgebra-isomorphism and $\Phi^*\beta = \alpha$. An NHM Hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ is now replaced by a *generalized NHM Hamiltonian system* $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \mathcal{X}, \omega, H)$.

We shall use the generalized symplectic structures in the next subsection and in paper II where we shall employ the pairs $(\mathcal{A}, \mathcal{X})$ with $\mathcal{X} = \text{ISDer}(\mathcal{A})$ to define quantum symplectic structure on superalgebras admitting outer as well as inner superderivations.

H. Augmented symplectics including ‘time’; the noncommutative analogue of Poincaré-Cartan form

We shall now augment the kinematic framework of NHM by including the evolution parameter t (‘time’) and obtain the non-commutative analogues of the Poincaré-Cartan form and of the symplectic version of Noether’s theorem (Souriau [68]).

For a system S with associated symplectic superalgebra (\mathcal{A}, ω) we construct the *extended system algebra* $\mathcal{A}^e = C^\infty(\mathbb{R}) \otimes \mathcal{A}$ (where the real line \mathbb{R} is the carrier space of ‘time’ t) whose elements are finite sums $\sum_i f_i \otimes A_i$ (with $f_i \in C^\infty(\mathbb{R}) \equiv \mathcal{A}_0$) which may be written as $\sum_i f_i A_i$. This algebra is the analogue of the algebra of functions on the ‘evolution space’ of Souriau (the Cartesian product of the time axis and the phase space — often referred to as the phase bundle). The superscript e in \mathcal{A}^e , may, therefore, also be taken to refer to ‘evolution’.

Derivations on \mathcal{A}_0 are of the form $g(t)\frac{d}{dt}$ and one-forms of the form $h(t)dt$ where g and h are smooth functions; there are no nonzero higher order forms. We have, of course, $dt(\frac{d}{dt}) = 1$.

A (super-)derivation D_1 on \mathcal{A}_0 and D_2 on \mathcal{A} extend trivially to (super-)derivations on \mathcal{A}^e as $D_1 \otimes id_{\mathcal{A}}$ and $id_{\mathcal{A}_0} \otimes D_2$ respectively; these trivial extensions may be informally denoted as D_1 and D_2 . With $f \otimes A$ written as fA , we can write $D_1(fA) = (D_1f)A$ and $D_2(fA) = f(D_2A)$.

Employing the mapping $\Xi : \mathcal{A} \rightarrow \mathcal{A}^e$ given by $\Xi(A) = 1 \otimes A$, which is an isomorphism of the algebra \mathcal{A} onto the subalgebra $\tilde{\mathcal{A}} \equiv 1 \otimes \mathcal{A}$ of \mathcal{A}^e , we write, for a p-form α on \mathcal{A} , $(\Xi^{-1})^*(\alpha) = \tilde{\alpha}$ and extend this form on $\tilde{\mathcal{A}}$ to one on \mathcal{A}^e by defining $\tilde{\alpha}(\frac{d}{dt}, \dots) = 0$. Similarly, we may extend the one-form dt on \mathcal{A}_0 to one on \mathcal{A}^e by defining $(dt)(X) = 0$ for all $X \in SDer(\tilde{\mathcal{A}})$.

The symplectic structure ω on \mathcal{A} induces, on \mathcal{A}^e , a generalized symplectic structure (of the type introduced in the previous subsection) with the distinguished Lie sub-superalgebra \mathcal{X} of $SDer(\mathcal{A}^e)$ taken to be the one consisting of the objects $\{id_{\mathcal{A}_0} \otimes D; D \in SDer(\mathcal{A})\}$ which constitute a Lie sub-superalgebra of $SDer(\mathcal{A}^e)$ isomorphic to $SDer(\mathcal{A})$, thus giving a generalized symplectic superalgebra $(\mathcal{A}^e, \mathcal{X}, \tilde{\omega})$. The corresponding PBs on \mathcal{A}^e are trivial extensions of those on \mathcal{A} obtained by treating the ‘time’ t as an external parameter; this amounts to extending the \mathbb{C} -linearity of PBs on \mathcal{A} to what is essentially \mathcal{A}_0 -linearity :

$$\{fA + gB, hC\}_{\mathcal{A}^e} = fh\{A, C\}_{\mathcal{A}} + gh\{B, C\}_{\mathcal{A}}$$

where, for clarity, we have put subscripts on the PBs to indicate the underlying superalgebras. We shall often drop these subscripts; the underlying (super-)algebra will be clear from the context.

To describe dynamics in \mathcal{A}^e , we extend the one-parameter family Φ_t of canonical transformations on \mathcal{A} generated by a Hamiltonian $H \in \mathcal{A}$ to a one-parameter family Φ_t^e of transformations on \mathcal{A}^e (which are ‘canonical’ in a certain sense, as we shall see below) given by

$$\Phi_t^e(fA) \equiv (fA)(t) = f(t)A(t) \equiv (\Phi_t^{(0)}f)\Phi_t(A)$$

where $\Phi_t^{(0)}$ is the one-parameter group of translations in ‘time’ whose action on \mathcal{A}_0 is given by $(\Phi_t^{(0)}f)(s) = f(s+t)$; extension of the action of Φ_t^e to general elements of \mathcal{A}^e is obtained by linearity. An infinitesimal transformation under the evolution Φ_t^e is given by

$$\begin{aligned} \delta(fA)(t) &\equiv (fA)(t + \delta t) - (fA)(t) \\ &= [\frac{df}{dt}A + f\{H, A\}_{\mathcal{A}}]\delta t \equiv \hat{Y}_H(fA)\delta t \end{aligned}$$

where

$$\hat{Y}_H = \frac{\partial}{\partial t} + \tilde{Y}_H. \quad (69)$$

Here $\frac{\partial}{\partial t}$ is the derivation on \mathcal{A}^e corresponding to the derivation $\frac{d}{dt}$ on \mathcal{A}_0 and [treating \mathcal{A} as a subalgebra of \mathcal{A}^e to replace $\{H, A\}_{\mathcal{A}}$ by $\{H, A\}_{\mathcal{A}^e}$]

$$\tilde{Y}_H = \{H, \cdot\}_{\mathcal{A}^e}. \quad (70)$$

Note that

- (i) $dt(\hat{Y}_H) = 1$;
- (ii) the right hand side of Eq.(70) (where H should be understood as $\tilde{H} = 1 \otimes H \in \mathcal{A}^e$) remains well defined if H is a general element of \mathcal{A}^e ('time dependent' Hamiltonian). Henceforth, in this subsection, H will be understood to be a general element of \mathcal{A}^e .

Writing, for $F \in \mathcal{A}^e$, $\Phi_t^e(F) = F(t)$, the obvious generalization of the NHM Hamilton's equation (49) to \mathcal{A}^e is the equation

$$\frac{dF(t)}{dt} = \hat{Y}_H F(t) = \frac{\partial F(t)}{\partial t} + \{H(t), F(t)\}. \quad (71)$$

We next consider an object in \mathcal{A}^e which contains complete information about the symplectic structure *and* dynamics [i.e. about $\tilde{\omega}$ and H (up to an additive constant multiple of I)] and is canonically determined by these objects. It is the 2-form

$$\Omega = \tilde{\omega} + dt \wedge dH. \quad (72)$$

Here d is the exterior derivative in \mathcal{A}^e induced by the exterior derivatives d_1 in \mathcal{A}_0 and d_2 in \mathcal{A} according to Eq.(28) and t is treated as an element $t \otimes I$ of \mathcal{A}^e so that $dt = d_1 t \otimes I$ where I is the identity element of \mathcal{A} . Noting that, by Eq.(28), the form $\tilde{\omega} = 1 \otimes \omega$ is closed, the form Ω is easily seen to be closed. If the symplectic structure on \mathcal{A} is exact (with $\omega = d_2 \theta$), we have $\Omega = d\Theta$ where

$$\Theta = \tilde{\theta} - H dt \quad (73)$$

is the NHM analogue of the Poincaré-Cartan form in classical mechanics.

Given $H = \sum_i f_i H_i$, we have

$$dH = \sum_i (d_1 f_i) H_i + \sum_i f_i (d_2 H_i) \equiv \tilde{d}_1 H + \tilde{d}_2 H.$$

Note that (a) since

$$\begin{aligned} \tilde{d}_1 H &= \sum_i (d_1 f_i) H_i = \sum_i \left(\frac{\partial f_i}{\partial t} dt \right) \otimes H_i = \sum_i \left(\frac{\partial f_i}{\partial t} \otimes H_i \right) (dt \otimes 1) \\ &= \frac{\partial H}{\partial t} dt, \end{aligned}$$

we have $dt \wedge dH = dt \wedge \tilde{d}_2 H$;

(b) since the symplectic superalgebra $(\tilde{\mathcal{A}}, \tilde{\omega})$ is isomorphic to (\mathcal{A}, ω) and since the functions f_i in H act essentially as numbers in the context of the symplectics at hand, we must have

$$i_{\tilde{Y}_H} \tilde{\omega} = -\tilde{d}_2 H.$$

The closed form Ω is generally not non-degenerate. It defines what may be called a *presymplectic structure* (Souriau [68]) on \mathcal{A}^e . In fact, we have here the noncommutative analogue of a special type of presymplectic structure called *contact structure* (Abraham and Marsden [1]; Berndt [8]); it essentially means that the presymplectic form is minimally degenerate.

A *symplectic action* of a Lie group G on the presymplectic space (\mathcal{A}^e, Ω) is the assignment, to every $g \in G$, an automorphism $\Phi(g)$ of the superalgebra \mathcal{A}^e having the usual group action properties and such that

- (i) $\Phi(g)^* \Omega = \Omega$;
- (ii) $\Phi(g) \tilde{\mathcal{A}}_0 \subset \tilde{\mathcal{A}}_0$ where $\tilde{\mathcal{A}}_0 = \mathcal{A}_0 \otimes I$.

The condition (i) implies, as in section III E, that, to every element ξ of the Lie algebra \mathcal{G} of G , corresponds a derivation \hat{Z}_ξ of \mathcal{A}^e such that $L_{\hat{Z}_\xi} \Omega = 0$ which, in view of the condition $d\Omega = 0$, is equivalent to the condition

$$d(i_{\hat{Z}_\xi} \Omega) = 0. \quad (74)$$

We shall now verify that the one-parameter family Φ_t^e of transformations on \mathcal{A}^e is symplectic/canonical. Time translations employed in the description of dynamics clearly satisfy the condition (ii). To verify (i), it is adequate to verify that Eq.(74) holds with $\hat{Z}_\xi = \hat{Y}_H$. We have, in fact, much more :

Proposition 3.6. *In the notation employed above, we have*

$$i_{\hat{Y}_H} \Omega = 0. \quad (75)$$

Proof. We have

$$\begin{aligned} i_{\hat{Y}_H} \Omega &= i_{\partial/\partial t} \Omega + i_{\tilde{Y}_H} \Omega \\ &= i_{\partial/\partial t} (dt \wedge \tilde{d}_2 H) + i_{\tilde{Y}_H} \tilde{\omega} + i_{\tilde{Y}_H} (dt \wedge \tilde{d}_2 H) \\ &= \tilde{d}_2 H - \tilde{d}_2 H - i_{\tilde{Y}_H} (\tilde{d}_2 H) dt \\ &= [i_{\tilde{Y}_H} (i_{\tilde{Y}_H} \tilde{\omega})] dt = 0. \quad \square \end{aligned}$$

A symplectic G -action (in the present context) is said to be *hamiltonian* if the 1-forms $i_{\hat{Z}_\xi} \Omega$ are exact, i.e. to each $\xi \in \mathcal{G}$, corresponds a ‘hamiltonian’ $\hat{h}_\xi \in \mathcal{A}^e$ (unique up to an additive constant multiple of the unit element) such that

$$i_{\hat{Z}_\xi} \Omega = -d\hat{h}_\xi. \quad (76)$$

Theorem (1) (Noncommutative symplectic version of Noether’s theorem). *Given the presymplectic space (\mathcal{A}^e, Ω) associated with the symplectic algebra (\mathcal{A}, ω) and the Hamiltonian $H \in \mathcal{A}^e$, the ‘hamiltonians’ (Noether invariants) \hat{h}_ξ corresponding to the Hamiltonian action of a connected Lie group G as in Eq.(76) are constants of motion.*

Proof. We have

$$\begin{aligned} \frac{d\hat{h}_\xi(t)}{dt} &= \hat{Y}_H(\hat{h}_\xi(t)) = (d\hat{h}_\xi)(\hat{Y}_H)(t) \\ &= -(i_{\hat{Z}_\xi} \Omega)(\hat{Y}_H)(t) = 0 \end{aligned} \quad (77)$$

where, in the last step, Eq.(75) has been used. \square

Some concrete examples of Noether invariants will be given in paper II.

IV. INTERACTING SYSTEMS IN NONCOMMUTATIVE HAMILTONIAN MECHANICS

We shall now consider the interaction of two systems S_1 and S_2 described individually as the NHM Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ ($i=1,2$) and treat the coupled system $S_1 + S_2$ also as an NHM Hamiltonian system. To facilitate this, we must obtain the relevant mathematical objects for the coupled system. The system algebra for the coupled system will be taken to be the (skew) tensor product $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$. We next consider the symplectic structure on \mathcal{A} .

A. The symplectic form and Poisson bracket on the (skew) tensor product of two symplectic superalgebras

We shall freely use the notations and constructions in section II E.

Given the symplectic forms $\omega^{(i)}$ on $\mathcal{A}^{(i)}$ [with PBs $\{, \}_i$ ($i=1,2$)] we explore the possibility of constructing a (preferably unique) *canonically induced*

symplectic form ω on \mathcal{A} . The term ‘canonically induced’ means that the construction of ω should use nothing besides the algebraic and symplectic structures on the superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$. [*Note.* No fundamental principle is violated if ω is taken to depend on objects representing the interaction between the two systems. In the formalism of NHM as developed above, however, the burden of the description of dynamics is carried by the Hamiltonian in the kinematic framework provided by the symplectic structure; we shall, therefore, not consider this more general possibility in this paper.] To ensure this, we impose on ω (apart from the obvious condition that it must be a symplectic form) the following conditions :

- (a) It should not depend on anything other than the objects $\omega^{(i)}$ and $I_{(i)}$ ($i=1,2$). (Note that the unit elements are the only distinguished elements of the algebras being considered.)
- (b) The restrictions of ω to $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$ be, respectively, $\omega^{(1)} \otimes I_2$ and $I_1 \otimes \omega^{(2)}$.

Recalling the equation (*) in section 2.5, these requirements lead to the unique choice

$$\omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}. \quad (78)$$

To verify that it is a symplectic form, we must show that it is (i) closed and (ii) nondegenerate. Eq.(28) gives

$$d\omega = (d_1\omega^{(1)}) \otimes I_2 + \omega^{(1)} \otimes d_2(I_2) + d_1(I_1) \otimes \omega^{(2)} + I_1 \otimes d_2\omega^{(2)} = 0$$

showing that ω is closed. It follows, therefore, that, on \mathcal{A} , a canonically induced presymplectic structure given by the presymplectic form (78) always exists. To show the nondegeneracy of ω , it is necessary and sufficient to show that, given $A \otimes B \in \mathcal{A}$ (with A and B homogeneous), there exists a unique homogeneous superderivation $Y = Y_{A \otimes B}$ in $SDer(\mathcal{A})$ such that

$$\begin{aligned} i_Y \omega = -d(A \otimes B) &= -(d_1 A) \otimes B - A \otimes d_2 B \\ &= i_{Y_A^{(1)}} \omega^{(1)} \otimes B + A \otimes i_{Y_B^{(2)}} \omega^{(2)} \end{aligned} \quad (79)$$

where $Y_A^{(1)}$ and $Y_B^{(2)}$ are the Hamiltonian superderivations associated with $A \in \mathcal{A}^{(1)}$ and $B \in \mathcal{A}^{(2)}$.

In the following developments, we shall denote the multiplication operators in $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and \mathcal{A} by μ_1 , μ_2 and μ respectively.

Proposition 4.1. *A linear mapping Y of $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ into itself is a homogeneous superderivation satisfying Eq.(79) if and only if (i) it is expressible as*

$$Y = Y_A^{(1)} \otimes \Psi_B^{(2)} + \Psi_A^{(1)} \otimes Y_B^{(2)} \quad (80)$$

where $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ are linear mappings (on $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively into themselves) satisfying the conditions $\Psi_A^{(1)}(I_1) = A$ and $\Psi_B^{(2)}(I_2) = B$, and (ii) it satisfies the relation (for homogeneous C and D)

$$Y \circ \mu(C \otimes D) - \eta_{Y, C \otimes D} \mu(C \otimes D) \circ Y = \mu(Y(C \otimes D)). \quad (81)$$

Proof. If : Equations (78) and (80) give

$$i_Y \omega = i_{Y_A^{(1)} \omega^{(1)}} \otimes \Psi_B^{(2)}(I_2) + \Psi_A^{(1)}(I_1) \otimes i_{Y_B^{(2)} \omega^{(2)}} = \text{right hand side of (79)}.$$

Eq.(81) guarantees that Y is a superderivation of \mathcal{A} .

Only if : To be a superderivation, Y must satisfy, by proposition (2.1), Eq.(81). Let $X \in SDer(\mathcal{A})$ such that $i_X \omega = i_Y \omega$ where Y now stands for the right hand side of (80). By proposition 2.1, X satisfies Eq.(81). We, therefore, need only show that $Z \equiv X - Y = 0$. Let $Z_1, Z_2 \in SDer(\mathcal{A})$ such that $Z = Z_1 + Z_2$ is the unique decomposition of Z in accordance with proposition 2.9; it means that

- (i) Z_1 equals Z on $\tilde{\mathcal{A}}^{(1)}$ and 0 on $\tilde{\mathcal{A}}^{(2)}$;
- (ii) Z_2 equals 0 on $\tilde{\mathcal{A}}^{(1)}$ and Z on $\tilde{\mathcal{A}}^{(2)}$.

We shall now prove, exploiting the non-degeneracy of $\tilde{\omega}^{(1)} = \omega^{(1)} \otimes I_2$ and $\tilde{\omega}^{(2)} = I_1 \otimes \omega^{(2)}$ (as symplectic forms on $\tilde{\mathcal{A}}^{(1)}$ and $\tilde{\mathcal{A}}^{(2)}$ respectively), that $Z_1 = 0 = Z_2$. We have

$$0 = i_Z \omega = i_Z(\tilde{\omega}^{(1)} + \tilde{\omega}^{(2)})$$

which gives

$$i_{Z_1} \tilde{\omega}^{(1)} = i_{Z_2} \tilde{\omega}^{(1)} = -i_{Z_1} \tilde{\omega}^{(2)} = -i_{Z_2} \tilde{\omega}^{(2)}.$$

Now $i_{Z_1} \tilde{\omega}^{(1)}$ is a one-form on $\tilde{\mathcal{A}}^{(1)}$ (hence of the form $\alpha \otimes I_2$ where α is a one-form on $\mathcal{A}^{(1)}$) whereas $-i_{Z_2} \tilde{\omega}^{(2)}$ is a one-form on $\tilde{\mathcal{A}}^{(2)}$ (hence of the form $I_1 \otimes \beta$ where β is a one-form on $\mathcal{A}^{(2)}$); the two can be equal (as forms on \mathcal{A}) only when $\alpha = 0 = \beta$. Nondegeneracy of $\tilde{\omega}^{(1)}$ and $\tilde{\omega}^{(2)}$ now implies $Z_1 = 0 = Z_2$. \square

The following developments explore the consequences of equations (80) and (81) leading ultimately to theorem (2).

Noting that $\mu(C \otimes D) = \mu_1(C) \otimes \mu_2(D)$, Eq.(81) with Y of Eq.(80) gives

$$\begin{aligned} & \eta_{BC} \{ [Y_A^{(1)} \circ \mu_1(C)] \otimes [\Psi_B^{(2)} \circ \mu_2(D)] + [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \circ \mu_2(D) \} \\ & - (-1)^\epsilon \{ [\mu_1(C) \circ Y_A^{(1)}] \otimes [\mu_2(D) \circ \Psi_B^{(2)}] + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes [\mu_2(D) \circ Y_B^{(2)}] \} \\ & = \eta_{BC} [\mu_1(\{A, C\}_1) \otimes \mu_2(\Psi_B^{(2)}(D)) + \mu_1(\Psi_A^{(1)}(C)) \otimes \mu_2(\{B, D\}_2)] \quad (82) \end{aligned}$$

where $\epsilon \equiv \epsilon_A \epsilon_C + \epsilon_B \epsilon_D + \epsilon_B \epsilon_C$ and we have used the relations $Y_A^{(1)}(C) = \{A, C\}_1$ and $Y_B^{(2)}(D) = \{B, D\}_2$.

The objects $Y_A^{(1)}$ and $Y_B^{(2)}$, being superderivations, satisfy relations of the form (1) :

$$\begin{aligned} Y_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ Y_A^{(1)} &= \mu_1(Y_A^{(1)}(C)) = \mu_1(\{A, C\}_1) \\ Y_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ Y_B^{(2)} &= \mu_2(\{B, D\}_2). \quad (83) \end{aligned}$$

Putting $D = I_2$ in Eq.(82), we have [noting that $\mu_2(D) = \mu_2(I_2) = id_{\mathcal{A}^{(2)}}$, and $\{B, I_2\}_2 = Y_B^{(2)}(I_2) = 0$]

$$\begin{aligned} [Y_A^{(1)} \circ \mu_1(C)] \otimes \Psi_B^{(2)} + [\Psi_A^{(1)} \circ \mu_1(C)] \otimes Y_B^{(2)} \\ - \eta_{AC} \{ [\mu_1(C) \circ Y_A^{(1)}] \otimes \Psi_B^{(2)} + [\mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)} \} \\ = \mu_1(\{A, C\}_1) \otimes \mu_2(B) \quad (84) \end{aligned}$$

which, along with equations (83), gives

$$\begin{aligned} \mu_1(\{A, C\}_1) \otimes [\Psi_B^{(2)} - \mu_2(B)] &= \\ - [\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ \Psi_A^{(1)}] \otimes Y_B^{(2)}. \quad (85) \end{aligned}$$

Similarly, putting $C = I_1$ in Eq.(82), we get

$$\begin{aligned} [\Psi_A^{(1)} - \mu_1(A)] \otimes \mu_2(\{B, D\}_2) &= \\ - Y_A^{(1)} \otimes [\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ \Psi_B^{(2)}]. \quad (86) \end{aligned}$$

Now, a careful consideration of equations (86) and (85) (which hold for arbitrary homogeneous elements $A, C \in \mathcal{A}^{(1)}$ and $B, D \in \mathcal{A}^{(2)}$) gives the relations (see appendix)

$$\Psi_A^{(1)} - \mu_1(A) = \lambda_1 Y_A^{(1)} \text{ for all } A \in \mathcal{A}^{(1)} \quad (87)$$

$$\Psi_B^{(2)} \circ \mu_2(D) - \eta_{BD} \mu_2(D) \circ \Psi_B^{(2)} = -\lambda_1 \mu_2(\{B, D\}_2) \quad (88)$$

$$\Psi_B^{(2)} - \mu_2(B) = \lambda_2 Y_B^{(2)} \text{ for all } B \in \mathcal{A}^{(2)} \quad (89)$$

$$\Psi_A^{(1)} \circ \mu_1(C) - \eta_{AC} \mu_1(C) \circ \Psi_A^{(1)} = -\lambda_2 \mu_1(\{A, C\}_1) \quad (90)$$

where λ_1 and λ_2 are complex numbers (the possibility of either or both being zero included). The restriction on A and B of being homogeneous has been removed in equations (87) and (89) because these equations do not involve parities explicitly and have both sides linear in A and B respectively.

Substituting for $\Psi_A^{(1)}$ and $\Psi_B^{(2)}$ from equations (87) and (89) in Eq.(80), we get

$$\begin{aligned} Y \equiv Y_{A \otimes B} &= Y_A^{(1)} \otimes [\mu_2(B) + \lambda_2 Y_B^{(2)}] + [\mu_1(A) + \lambda_1 Y_A^{(1)}] \otimes Y_B^{(2)} \\ &= Y_A^{(1)} \otimes \mu_2(B) + \mu_1(A) \otimes Y_B^{(2)} + (\lambda_1 + \lambda_2) Y_A^{(1)} \otimes Y_B^{(2)}. \end{aligned} \quad (91)$$

Note that only the combination $(\lambda_1 + \lambda_2) \equiv \lambda$ appears in Eq.(91). To have a unique Y, we must obtain an equation fixing λ in terms of given quantities.

Substituting for $\Psi_A^{(1)}$ from Eq.(87) into Eq.(90) and using the first of equations (83), we have

$$\lambda \mu_1(\{A, C\}_1) = -\mu_1([A, C]) \text{ for all } A, C \in \mathcal{A}^{(1)} \quad (92)$$

[argument for the phrase ‘for all’ in Eq.(92) being the same as for the equations (87) and (89)] which implies

$$\lambda \{A, C\}_1 = -[A, C] + K_1(A, C) \quad (93)$$

where $K_1(A, C)$ is an element of $\mathcal{A}^{(1)}$ such that

$$K_1(A, C)E = 0 \text{ for all } E \in \mathcal{A}^{(1)}.$$

Since $\mathcal{A}^{(1)}$ is unital, we must have $K_1 = 0$. Proceeding similarly with the equations (89) and (88), we have, therefore,

$$\lambda \{A, C\}_1 = -[A, C] \text{ for all } A, C \in \mathcal{A}^{(1)} \quad (94)$$

$$\lambda \{B, D\}_2 = -[B, D] \text{ for all } B, D \in \mathcal{A}^{(2)}. \quad (95)$$

We have not one but two equations of the type we have been looking for. This is a signal for the emergence of nontrivial conditions (for the desired symplectic structure on the tensor product superalgebra to exist).

Let us consider these equations for the various possible situations (corresponding to whether or not one or both the superalgebras are supercommutative) :

(i) Let $\mathcal{A}^{(1)}$ be supercommutative. Since the PB coming from a symplectic structure must be nontrivial (i.e. not identically zero), Eq.(94) implies that $\lambda = 0$. Eq.(95) then implies that $\mathcal{A}^{(2)}$ must also be supercommutative. It follows that

- (a) when both the superalgebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are supercommutative, the unique Y is given by Eq.(91) with $\lambda(= \lambda_1 + \lambda_2) = 0$;
- (b) when one of them is supercommutative and the other is not, the ω of Eq.(78) does not define a symplectic structure on \mathcal{A} .

(ii) Let the superalgebra $\mathcal{A}^{(1)}$ be non-supercommutative. Eq.(94) then implies that $\lambda \neq 0$, which, along with Eq.(95) implies that the superalgebra $\mathcal{A}^{(2)}$ is also non-supercommutative [which is also expected from (b) above]. We now have

$$\{A, C\}_1 = -\lambda^{-1}[A, C], \quad \{B, D\}_2 = -\lambda^{-1}[B, D]. \quad (96)$$

Equations (96) imply that, when both the superalgebras are non-supercommutative, a ‘canonically induced’ symplectic structure on their (skew) tensor product exists if and only if each superalgebra has a quantum symplectic structure with the *same* parameter $(-\lambda)$, i.e.

$$\omega^{(1)} = -\lambda\omega_c^{(1)}, \quad \omega^{(2)} = -\lambda\omega_c^{(2)} \quad (97)$$

where $\omega_c^{(i)}$ (i=1,2) are the canonical symplectic forms on the two superalgebras. Reality of these symplectic forms requires λ to be pure imaginary ($\lambda = ih_0$ with h_0 real). The traditional quantum symplectic structure is obtained with $h_0 = \hbar$ [see Eq.(44)].

In all the permitted cases, the PB on the superalgebra $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ is given by

$$\begin{aligned} \{A \otimes B, C \otimes D\} = Y_{A \otimes B}(C \otimes D) &= \eta_{BC}[\{A, C\}_1 \otimes BD + AC \otimes \{B, D\}_2 \\ &\quad + \lambda\{A, C\}_1 \otimes \{B, D\}_2] \end{aligned} \quad (98)$$

where the parameter λ vanishes in the super-commutative case; in the non-supercommutative case, it is the universal parameter appearing in the symplectic forms (97).

Noting that, in both the permitted cases,

$$\begin{aligned}\lambda\{A, C\}_1 \otimes \{B, D\}_2 &= -[A, C] \otimes \{B, D\}_2 = -\{A, C\}_1 \otimes [B, D] \\ &= -\frac{1}{2}[A, C] \otimes \{B, D\}_2 - \frac{1}{2}\{A, C\}_1 \otimes [B, D],\end{aligned}$$

the PB of Eq.(98) can be written in the more symmetric form

$$\begin{aligned}\{A \otimes B, C \otimes D\} \\ = \eta_{BC} \left[\{A, C\}_1 \otimes \frac{BD + \eta_{BD}DB}{2} + \frac{AC + \eta_{AC}CA}{2} \otimes \{B, D\}_2 \right] \quad (99)\end{aligned}$$

To close the argument, we must check whether or not we have exhausted the contents of Eq.(82) (recall that we have used only the two special cases of this equation corresponding to $D = I_2$ and $C = I_1$). Let $Y = Y_1 + Y_2$ be the (unique) breakup of Y in accordance with the proposition 2.9. By putting $D = I_2$ in Eq.(82), one is, in fact, considering Y_1 ; similarly, putting $C = I_1$ amounts to considering Y_2 . Exploring the implications of Eq.(82) for Y_1 and Y_2 amounts to doing the same for Y . This equation, therefore, cannot have any additional implications for Y .

We have proved the following theorem.

Theorem (2) (Commutative-noncommutative non-interaction and universality of quantum symplectic structure). *(a) Given two symplectic superalgebras $(\mathcal{A}^{(i)}, \omega^{(i)})$ ($i=1,2$), the (skew) tensor product $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ always admits the ‘canonically induced’ presymplectic structure given by the 2-form (78);*

(b) it is a symplectic structure if and only if either (i) both the superalgebras are supercommutative, or (ii) both are non-supercommutative and have a quantum symplectic structure (in the sense of section III C) with a universal parameter $b = -ih_0$ where $h_0 \in \mathbb{R}$.

(c) In both the cases the PB on \mathcal{A} is given by Eq.(99).

Note. (i) The theorem says nothing about the sign of the universal real parameter h_0 (which will be later identified with the Planck constant \hbar).

(ii) The two forms $\omega^{(i)}$ ($i=1,2$) of Eq.(97) represent genuine symplectic structures only if the superalgebras $\mathcal{A}^{(i)}$ ($i=1,2$) are ‘special’ (i.e. have only inner superderivations; see section III C). Since the initial objects $\mathcal{A}^{(i)}$ ($i=1,2$) were assumed to be symplectic superalgebras, an implicit conclusion of the theorem is that, in the noncommutative case, these superalgebras must be special. A more general version of the theorem is obtained by replacing, in part (a), the symplectic superalgebras $(\mathcal{A}^{(i)}, \omega^{(i)})$ by generalized symplectic superalgebras $(\mathcal{A}^{(i)}, \mathcal{X}^{(i)}, \omega^{(i)})$ ($i=1,2$); then, part (b) (ii) will take the modified form : ‘both are non-supercommutative with $\mathcal{X}^{(i)} = ISDer(\mathcal{A}^{(i)})$ and have a generalized quantum symplectic structure ... $h_0 \in \mathbb{R}$.’

(iii) The non-super version of Eq.(99) was [wrongly, not realizing that Y of Eq.(80) is not always a (super-)derivation] put forward by the author as the PB for a tensor product of algebras in the general case in (Dass [23]). M.J.W. Hall pointed out to the author (private communication) that it does not satisfy Jacobi identity in some cases, as shown, for example, in (Caro and Salcedo [14]). Revised calculations by the author then led to the results presented above.

(iv) After the first version of the present work appeared in the arXiv [0909.4606 v1 (math-ph)], L.L. Salcedo drew the author’s attention (private communication) to the paper (Salcedo [62]) in which similar results are obtained in an algebraic treatment of systems employing a finite number of q-p pairs.

B. Dynamics of interacting systems

Given the individual systems S_1 and S_2 as the NHM Hamiltonian systems $(\mathcal{A}^{(i)}, \omega^{(i)}, H^{(i)})$ ($i = 1,2$) where the two superalgebras are either both supercommutative or both non-supercommutative, the coupled system $(S_1 + S_2)$ has associated with it a symplectic superalgebra (\mathcal{A}, ω) where $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$ and ω is given by Eq.(78). We form an NHM Hamiltonian system (\mathcal{A}, ω, H) with the Hamiltonian H given by

$$H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int} \quad (100)$$

where the interaction Hamiltonian is generally of the form

$$H_{int} = \sum_{i=1}^n F_i \otimes G_i.$$

The evolution (in the Heisenberg type picture) of an observable $A(t) \otimes B(t)$

of the coupled system is governed by the NHM Hamilton's equation

$$\begin{aligned}\frac{d}{dt}[A(t) \otimes B(t)] &= \{H, A(t) \otimes B(t)\} \\ &= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2 \\ &\quad + \{H_{int}, A(t) \otimes B(t)\}.\end{aligned}\tag{101}$$

where we have used equations (49), (100) and (99). In the Schrödinger type picture, the time evolution of states of the coupled system is given by the NHM Liouville equation (51) with the Hamiltonian of Eq.(100). In favorable situations, the NHM evolution equations for observables and states may be written for finite time intervals by using appropriate exponentiations of the evolution operators in equations (49) and (51).

The main lesson from this section is that *all* systems in nature *whose interaction with other systems can be talked about* must belong to only one of the two ‘worlds’ : the ‘commutative world’ in which all system superalgebras are super-commutative and the ‘noncommutative world’ in which all system superalgebras are non-supercommutative with a *universal* (generalized) quantum symplectic structure. In view of the familiar inadequacy of the commutative/classical physics, the ‘real’ world must clearly be the noncommutative world; its systems (satisfying an extra condition of mutual compatibility between observables and pure states to be introduced in paper II) will be called *quantum systems*. The classical systems with commutative system algebras and traditional symplectic structures will appear only in the appropriately defined classical limit (or, more generally, in the classical approximation) of quantum systems.

V. CONCLUDING REMARKS

The development of NHM presented above constitutes the first concrete step towards the desired unification of probability theory and physics. The use of the observable-state language in an algebraic setting is, of course, not new; what is new is the unification of noncommutative *symplectic* geometry with noncommutative probability in such a setting. As we shall see in the next and later papers in this series, this makes possible a much better use of the true potential of the algebraic formalism.

Emergence of a natural place for the Planck constant in the formalism (Theorem 2) appears to indicate quite strongly that we are on the right track.

The absence of quantum-classical interaction (theorem 2) is, in fact, a non-problem in NHM (or its augmented version, supmech, described in paper II). Given a quantum system S_1 and a classical system S_2 , one can treat them as quantum systems, work in NHM employing the formalism of section IV B and then take the classical (or semiclassical) approximation for S_2 [remaining all the time in NHM (recall that NHM has both quantum and classical mechanics as special subdisciplines)]. We shall see this idea at work in the treatment of quantum measurements in paper III.

The reader should note the caution exercised in framing the first sentence in the last para of the previous section (recall the phrase in *italics*). Theorem (2) permits the construction of an NHM-based formalism in which the universe as a whole is described classically but all its subsystems whose interactions with other systems can be talked about (i.e. meaningfully considered) are described quantum mechanically.

The present work has brought us on the threshold of a formalism that provides for an autonomous development of QM as a universal mechanics. Development of such a formalism will be the burden of paper II.

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APPENDIX : STUDY OF IMPLICATIONS OF THE EQUATIONS (85) AND (86)

Let $V^{(i)} = \mathcal{L}(\mathcal{A}^{(i)})$ ($i=1,2$) be the space of linear mappings of the super-algebra $\mathcal{A}^{(i)}$ into itself and $V = V^{(1)} \otimes V^{(2)}$. The equations (85) and (86) are equations in the space V . Eq.(85) is of the form

$$x(A, C) \otimes v(B) = y(A, C) \otimes w(B) \quad (102)$$

where A, C are arbitrary homogeneous elements in $\mathcal{A}^{(1)}$ and B is a general element of $\mathcal{A}^{(2)}$. The functions $x(\cdot, \cdot)$ and $y(\cdot, \cdot)$ are bilinear and skew-symmetric and the functions $v(\cdot)$ and $w(\cdot)$ are linear.

The most general implication of Eq.(102) can be expressed in the form of

the relations

$$a_1x = a_2y; \quad a_3v = a_4w \quad (103)$$

where the parameters $a_i \in \mathbb{C}$ ($i = 1, \dots, 4$) satisfy the condition

$$a_1a_3 = a_2a_4. \quad (104)$$

Now, the quantities $x(A, C) = \mu_1(\{A, C\}_1)$ and $w(B) = Y_B^{(2)}$ cannot be identically zero [this follows from the non-degeneracy of the symplectic forms of the superalgebras $\mathcal{A}^{(i)}$ ($i=1,2$)]. It follows that, in Eq.(103), we must have $a_2 \neq 0$, $a_3 \neq 0$. We now have, from Eq.(104)

$$\frac{a_1}{a_2} = \frac{a_4}{a_3} = a, \text{ say}$$

and, from Eq.(103)

$$y = ax, \quad v = aw. \quad (105)$$

Since the possibilities $y \equiv 0$ and $v \equiv 0$ cannot be ruled out, we must allow the value $a = 0$.

The equations (105) are precisely the equations (89) and (90) with $a = \lambda_2$. Similar arguments with Eq.(86) lead to the equations (87) and (88).

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